Application Moment et Réduction en Mécanique

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OVERVIEW OF THE COURSE

- **Symplectic manifolds**
- **Poisson manifolds**
- **Lie group actions**
- **Abstract symmetry reduction**

- **Cotangent bundle reduction**

- **Lagrangian approach to reduction**

- **Conservation laws via generalized distributions**

- **The optimal momentum map and groupoids**

- **Optimal reduction**

- **Singular point reduction**

- **Singular orbit reduction**

- **Poisson reduction**

- **Coisotropic reduction**

- **Cosymplectic reduction**

SYMPLECTIC MANIFOLDS

A **symplectic manifold** is a pair (M, ω), where M is a manifold and $\omega \in \Omega^2(M)$ is a closed non–degenerate two–form on M, that is,

- $\mathbf{d}\omega = 0$
- for every $m \in M$, the map

$$v \in T_m M \mapsto \omega(m)(v, \cdot) \in T_m^* M$$

is a linear isomorphism.

If ω is allowed to be degenerate, (M, ω) is called a **presymplectic manifold**. A **Hamiltonian dynamical system** is a triple (M, ω, h), where (M, ω) is a symplectic manifold and $h \in C^{\infty}(M)$ is the **Hamiltonian function** of the system. By non–degeneracy of the symplectic form ω, to each Hamiltonian system one can associate a **Hamiltonian vector field** $X_h \in \mathfrak{X}(M)$, defined by the equality

$$\mathbf{i}_{X_h}\omega := \omega(X_h, \cdot) = \mathbf{d}h.$$

Example V vector space, V^* its dual. Let $Z = V \times V^*$. The **canonical symplectic form** Ω on Z is defined by

$$\Omega((v_1, \alpha_1), (v_2, \alpha_2)) := \langle \alpha_2, v_1 \rangle - \langle \alpha_1, v_2 \rangle.$$

$$[\Omega] = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} =: \mathbb{J}$$

Example Q manifold, T^*Q its cotangent bundle, $\pi_Q : T^*Q \to Q$ projection. The **canonical one-form** Θ on T^*Q defined by

$$\Theta(\beta) \cdot v_\beta := \left\langle \beta, T_\beta \pi_Q \left(v_\beta \right) \right\rangle, \quad \beta \in T^*Q, \quad v_\beta \in T_\beta(T^*Q).$$

In canonical coordinates $\Theta = p_i dq^i$

The **canonical symplectic form** Ω on the cotangent bundle T^*Q is defined by $\Omega = -\mathbf{d}\Theta$.

Darboux theorem: Locally $\omega|_U = \sum_{i=1}^{n} \mathbf{d}q^i \wedge \mathbf{d}p_i$.

In canonical coordinates, X_h is determined by the well-known **Hamilton equations**,

$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i}.$$

The **Poisson bracket** of $f, g \in C^\infty(M)$ is the function $\{f,\, g\} \in C^\infty(M)$ defined by

$$\{f,\, g\}(z) = \omega(z)\left(X_f(z),\, X_g(z)\right).$$

In canonical coordinates, the Poisson bracket has the form

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right) .$$

POISSON MANIFOLDS

- $(M, \{\cdot,\cdot\})$ **Poisson manifold** if $(C^\infty(M), \{\cdot,\cdot\})$ Lie algebra such that

$$\{fg, h\} = f\{g, h\} + g\{f, h\}$$

- **Casimir functions** are the elements of the center of $(C^\infty(M), \{\cdot,\cdot\})$.

- **Hamiltonian vector field** of $h \in C^\infty(M)$

$$\boldsymbol{\pounds}_{X_h} f := \langle \mathbf{d}f, X_h \rangle := X_h[f] = \{f, h\}, \ \text{ for all } \ f \in C^\infty(M).$$

Example: The Lie-Poisson bracket. The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ is a Poisson manifold with respect to the **$\pm$-Lie–Poisson** brackets $\{\cdot,\cdot\}_\pm$ defined by

$$\{f,g\}_\pm(\mu) := \pm\left\langle \mu, \left[\frac{\delta f}{\delta\mu}, \frac{\delta g}{\delta\mu}\right]\right\rangle ,$$

where $\frac{\delta f}{\delta\mu} \in \mathfrak{g}$ is defined by

$$\left\langle \nu, \frac{\delta f}{\delta\mu}\right\rangle := Df(\mu)\cdot\nu,$$

for any $\nu \in \mathfrak{g}^*$. The Hamiltonian vector field of $h \in C^\infty(\mathfrak{g}^*)$ ($\dot{f} = \{f,h\} \Leftrightarrow X_h = \{\cdot, f\}$) is given by

$$X_h(\mu) = \mp\mathrm{ad}^*_{\delta h/\delta\mu}\mu, \quad \mu \in \mathfrak{g}^*.$$

Example: Frozen Lie-Poisson bracket. Same notations as before. Let $\nu \in \mathfrak{g}^*$ and define the **frozen Lie–Poisson** brackets $\{\cdot, \cdot\}_\pm$ defined by

$$\{f, g\}^\nu_\pm(\mu) := \pm \left\langle \nu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle .$$

The Hamiltonian vector field of $h \in C^\infty(\mathfrak{g}^*)$ is given by

$$X_h(\mu) = \mp \mathrm{ad}^*_{\delta h/\delta \mu} \nu, \quad \mu \in \mathfrak{g}^*.$$

The Lie-Poisson and frozen Lie-Poisson bracket are **compatible**, that is, $\{\,,\}_\pm + s\{\,,\}^\nu_\pm$ is also a Poisson bracket on $\mathfrak{g}^*$ for any $\nu \in \mathfrak{g}^*$ and any $s \in \mathbb{R}$.

Example: Operator Algebra Brackets. $\mathcal{H}$ be a complex Hilbert space.

- $\mathfrak{S}(\mathcal{H})$, **trace class operators**

- $\mathfrak{HS}(\mathcal{H})$, **Hilbert-Schmidt operators**

- $\mathfrak{K}(\mathcal{H})$, **compact operators**

- $\mathfrak{B}(\mathcal{H})$, **bounded operators**

They form involutive Banach algebras. $\mathfrak{S}(\mathcal{H})$, $\mathfrak{HS}(\mathcal{H})$, $\mathfrak{K}(\mathcal{H})$ are self adjoint ideals in $\mathfrak{B}(\mathcal{H})$.

$$\mathfrak{S}(\mathcal{H}) \subset \mathfrak{HS}(\mathcal{H}) \subset \mathfrak{K}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{H})$$

$$\mathfrak{K}(\mathcal{H})^* \cong \mathfrak{S}(\mathcal{H}), \quad \mathfrak{HS}(\mathcal{H})^* \cong \mathfrak{HS}(\mathcal{H}), \quad \mathfrak{S}(\mathcal{H})^* \cong \mathfrak{B}(\mathcal{H});$$

the right hand sides are all Banach Lie algebras. These dualities are implemented by the strongly nondegenerate pairing

$$\langle x, \rho \rangle = \mathrm{trace}\,(x\rho)$$

where $x \in \mathfrak{S}(\mathcal{H})$, $\rho \in \mathfrak{K}(\mathcal{H})$ for the first isomorphism, $\rho, x \in \mathfrak{HS}(\mathcal{H})$ for the second isomorphism, and $x \in \mathfrak{B}(\mathcal{H})$, $\rho \in \mathfrak{S}(\mathcal{H})$ for the third isomorphism.

The Banach spaces $\mathfrak{S}(\mathcal{H})$, $\mathfrak{H}\mathfrak{S}(\mathcal{H})$, and $\mathfrak{K}(\mathcal{H})$ are Banach Lie-Poisson spaces in a rigorous functional analytic sense. The Lie-Poisson bracket becomes in this case

$$\{F, H\}(\rho) = \pm \operatorname{trace}\left([\mathbf{D}F(\rho), \mathbf{D}H(\rho)]\rho\right)$$

where ρ is an element of $\mathfrak{S}(\mathcal{H})$, $\mathfrak{H}\mathfrak{S}(\mathcal{H})$, or $\mathfrak{K}(\mathcal{H})$, respectively. The bracket $[\mathbf{D}F(\rho), \mathbf{D}H(\rho)]$ denotes the commutator bracket of operators. The Hamiltonian vector field associated to H is given by

$$X_H(\rho) = \pm[\mathbf{D}H(\rho), \rho].$$

The Poisson tensor. The derivation property of the Poisson bracket implies that for any two functions $f, g \in C^\infty(M)$, the value of the bracket $\{f, g\}(z)$ on f only through $\mathbf{d}f(z)$ which allows us to define a contravariant antisymmetric two-tensor $B \in \Lambda^2(M)$ by

$$B(z)(\alpha_z, \beta_z) = \{f, g\}(z),$$

with $\mathbf{d}f(z) = \alpha_z$ and $\mathbf{d}g(z) = \beta_z$. This tensor is called the **Poisson tensor** of M. The vector bundle map $B^\sharp : T^*M \to TM$ naturally associated to B is defined by

$$B(z)(\alpha_z, \beta_z) = \langle \alpha_z, B^\sharp(\beta_z) \rangle.$$

Its range $D := B^\sharp(T^*M) \subset TM$ is called the **characteristic distribution**. For any point $m \in M$, the dimension of $D(m)$ as a vector subspace of T_mM is called the **rank** of the Poisson manifold $(M, \{\cdot, \cdot\})$ at the point m.

The Weinstein coordinates of a Poisson manifold. Let $(M, \{\cdot,\cdot\})$ be a m–dimensional Poisson manifold and $z_0 \in M$ a point where the rank of $(M, \{\cdot,\cdot\})$ equals $2n$, $0 \leq 2n \leq m$. There exists a chart (U, φ) of M whose domain contains the point z_0 and such that the associated local coordinates, denoted by

$$(q^1, \ldots, q^n, p_1, \ldots, p_n, z^1, \ldots, z^{m-2n}),$$

satisfy

$$\{q^i, q^j\} = \{p_i, p_j\} = \{q^i, z^k\} = \{p_i, z^k\} = 0,$$

and $\{q^i, p_j\} = \delta^i_j$, for all i, j, k, $1 \leq i, j \leq n$, $1 \leq k \leq m-2n$.

For all k, l, $1 \leq k, l \leq m - 2n$, the Poisson bracket $\{z^k, z^l\}$ is a function of the local coordinates $z^1, \ldots, z^{m-2n}$ exclusively, and vanishes at z_0. Hence, the restriction of the bracket $\{\cdot, \cdot\}$ to the coordinates $z^1, \ldots, z^{m-2n}$ induces a Poisson structure that is usually referred to as the **transverse Poisson structure** of $(M, \{\cdot, \cdot\})$ at m.

If the rank is equal to $2n$ in a neighborhood of z_0, then the transverse structure is zero.

A smooth mapping $\varphi : (M_1, \{\cdot, \cdot\}_1) \to (M_2, \{\cdot, \cdot\}_2)$ is **canonical** or **Poisson** if for all $g, h \in C^\infty(M_2)$ we have

$$\varphi^*\{g, h\}_2 = \{\varphi^* g, \varphi^* g\}_1 \,.$$

In the symplectic category, $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ **canonical** or **symplectic** if

$$\varphi^* \omega_2 = \omega_1.$$

- Symplectic maps are immersions.

- A diffeomorphism $\varphi : M_1 \to M_2$ between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is symplectic if and only if it is Poisson.

- If the symplectic map $\varphi : M_1 \to M_2$ is not a diffeomorphism it may not be a Poisson map.

- A diffeomorphism $\varphi : T^*S \to T^*Q$ preserves the canonical one-forms Θ_Q on T^*Q and Θ_S on T^*S if and only if φ is the cotangent lift T^*f of some diffeomorphism $f : Q \to S$.

Proof Suppose that $f : Q \to S$ is a diffeomorphism. Then for $\beta \in T^*S$ and $v \in T_\beta(T^*S)$ we have

$$\begin{aligned}
\left(\left(T^*f\right)^* \Theta_Q\right)(\beta) \cdot v &= \Theta_Q\left(T^*f(\beta)\right) \cdot TT^*f(v) \\
&= \left\langle T^*f(\beta), \left(T\pi_Q \circ TT^*f\right)(v)\right\rangle \\
&= \left\langle \beta, T(f \circ \pi_Q \circ T^*f)(v)\right\rangle \\
&= \langle \beta, T\pi_S(v)\rangle
\end{aligned}$$

because $f \circ \pi_Q \circ T^*f = \pi_S$.

Idea for the converse. Assume that $\varphi^*\Theta_Q = \Theta_S$, i.e.,

$$\left\langle \varphi(\beta), T(\pi_Q \circ \varphi)(v)\right\rangle = \langle \beta, T\pi_S(v)\rangle, \quad \forall \beta \in T^*S,\ v \in T_\beta(T^*S)$$

Since φ is a diffeomorphism, the range of $T_\beta(\pi_Q \circ \varphi)$ is $T_{\pi_Q(\varphi(\beta))}Q$, so letting $\beta = 0 \Rightarrow \varphi(0) = 0$. Argue similarly for φ^{-1} and conclude that φ restricted to the zero section S of T^*S is a diffeomorphism onto the zero section Q of T^*Q. Define $f := \varphi^{-1}|Q$. Now one shows that φ is fiber preserving, i.e., $f \circ \pi_Q = \pi_S \circ \varphi^{-1}$. This is the main technical point. Then, using this, one shows that $\varphi = T^*f$. $\square$

Classical coordinate proof of the first part. Write

$$(s^1, \ldots, s^n) = f(q^1, \ldots, q^n)$$

Since $f : Q \to S$ is diffeomorphism, we can solve $q^i = q^i(s^1, \ldots, s^n)$. Coordinates on T^*Q are $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ and on T^*S they are $(s^1, \ldots, s^n, r_1, \ldots, r_n)$. So, both q^i and p_j are functions of $(s^1, \ldots, s^n, r_1, \ldots, r_n)$. The map T^*f is given by

$$T^*f(s^1, \ldots, s^n, r_1, \ldots, r_n) = (q^1, \ldots, q^n, p_1, \ldots, p_n).$$

But then, locally,

$$(\Theta_S =) r_i ds^i = r_i \frac{\partial s^i}{\partial q^k} dq^k = p_k dq^k \left(= (T^*f)^* \Theta_Q\right)$$

Let $(S, \{\cdot,\cdot\}^S)$ and $(M, \{\cdot,\cdot\}^M)$ be two Poisson manifolds such that $S \subset M$ and the inclusion $i_S : S \hookrightarrow M$ is an immersion. $(S, \{\cdot,\cdot\}^S)$ is a **Poisson submanifold** of $(M, \{\cdot,\cdot\}^M)$ if i_S is a canonical map.

An immersed submanifold Q of M is called a **quasi Poisson submanifold** of $(M, \{\cdot,\cdot\}^M)$ if for any $q \in Q$, any open neighborhood U of q in M, and any $f \in C^\infty_M(U)$ we have

$$X_f(i_Q(q)) \in T_q i_Q(T_q Q),$$

where $i_Q : Q \hookrightarrow M$ is the inclusion and X_f is the Hamiltonian vector field of f on U with respect to the restricted Poisson bracket $\{\cdot,\cdot\}^M_U$.

- On a quasi Poisson submanifold there is a unique Poisson structure that makes it into a Poisson submanifold.

- Any Poisson submanifold is quasi Poisson.

The converse is not true!

Counterexample. Let $(M = \mathbb{R}^2, B)$ where

$$B(x, y) = \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}$$

and $(Q = \mathbb{R}^2, \omega_{\text{can}})$. The identity map $\text{id} : Q \to M$ is obviously not a Poisson diffeomorphism because one structure has leaves and the other is non-degenerate. But is is also clear that any Hamiltonian vector field relative to B is tangent to $Q = \mathbb{R}^2$ and hence (Q, ω_{can}) is a quasi-Poisson submanifold of (M, B).

Given two symplectic manifolds (M,ω) and (S,ω_S) such that $S \subset M$ and the inclusion $i : S \hookrightarrow M$ is an immersion, the manifold (S,ω_S) is a **symplectic submanifold** of (M,ω) when i is a symplectic map.

Symplectic submanifolds of a symplectic manifold (M,ω) are in general neither Poisson nor quasi Poisson manifolds of M.

The only quasi Poisson submanifolds of a symplectic manifold are its open sets which are, in fact, Poisson submanifolds.

Symplectic Foliation Theorem. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and D the associated characteristic distribution. D is a smooth and integrable generalized distribution and its maximal integral leaves form a generalized foliation decomposing M into initial submanifolds $\mathcal{L}$, each of which is symplectic with the unique symplectic form that makes the inclusion $i : \mathcal{L} \hookrightarrow M$ into a Poisson map, that is, $\mathcal{L}$ is a Poisson submanifold of $(M, \{\cdot, \cdot\})$.

Example: Let $\mathfrak{g}^*$ with the Lie-Poisson structure. The symplectic leaves of the Poisson manifolds $(\mathfrak{g}^*, \{\cdot,\cdot\}_\pm)$ coincide with the connected components of the orbits of the elements in $\mathfrak{g}^*$ under the coadjoint action. In this situation, the symplectic form for the leaves is given by the **Kostant–Kirillov–Souriau (KKS)** or **orbit symplectic form**

$$\omega^{\pm}_{\mathcal{O}}(\nu)\left(-\operatorname{ad}^*_\xi \nu, -\operatorname{ad}^*_\eta \nu\right) = \pm \langle \nu, [\xi, \eta] \rangle .$$

- $(M, \{\cdot, \cdot\})$ Poisson manifold. G acts **canonically** on M when

$$\Phi_g^*\{f, h\} = \{\Phi_g^* f, \Phi_g^* h\}$$

for all $g \in G$.

- Easy Poisson reduction: $(M, \{\cdot, \cdot\})$ Poisson manifold, G Lie group acting canonically, freely, and properly on M. The orbit space M/G is a Poisson manifold with bracket

$$\{f, g\}^{M/G}(\pi(m)) = \{f \circ \pi, g \circ \pi\}(m)$$

- Reduction of Hamiltonian dynamics: $h \in C^\infty(M)^G$ reduces to $\overline{h} \in C^\infty(M/G)$ given by $\overline{h} \circ \pi = h$ such that

$$X_{\overline{h}} \circ \pi = T\pi \circ X_h$$

- What about the symplectic leaves? This is where symplectic reduction comes in.

- **Lie-Poisson reduction:** Left quotient $(T^*G)/G \cong \mathfrak{g}^*_-$. The map is: $[\alpha_g] \mapsto T_e^* R_g(\alpha_g)$. Direct proof. Discuss later. Notice that the quotient is for a *left* action and the map is given by *right* translation. Will be proved later.

LIE GROUP ACTIONS

M a manifold and G a Lie group. A **left action** of G on M is a smooth mapping $\Phi : G \times M \to M$ such that

(i) $\Phi(e, z) = z$, for all $z \in M$ and

(ii) $\Phi(g, \Phi(h, z)) = \Phi(gh, z)$ for all $g, h \in G$ and $z \in M$.

We will often write

$$g \cdot z := \Phi(g, z) := \Phi_g(z) := \Phi^z(g).$$

The triple (M, G, Φ) is called a **G-space** or a **G-manifold**.

Examples of group actions

- **Translation and conjugation.** The **left (right) translation** $L_g : G \to G$, $(R_g)\ h \mapsto gh$, induces a left (right) action of G on itself.

- The **inner automorphism** $\mathrm{AD}_g : G \to G$, given by $\mathrm{AD}_g := R_{g^{-1}} \circ L_g$ defines a left action of G on itself called **conjugation**.

- **Adjoint and coadjoint action.** The differential at the identity of the conjugation mapping defines a linear left action of G on $\mathfrak{g}$ called the **adjoint representation** of G on $\mathfrak{g}$

$$\mathrm{Ad}_g := T_e \, \mathrm{AD}_g : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

If $\mathrm{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual of Ad_g, then the map

$$\begin{array}{rccc} \Phi : & G \times \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^* \\ & (g, \nu) & \longmapsto & \mathrm{Ad}_{g^{-1}}^* \, \nu, \end{array}$$

defines also a linear left action of G on $\mathfrak{g}^*$ called the **coadjoint representation** of G on $\mathfrak{g}^*$.

- **Group representation.** If the manifold M is a vector space V and G acts linearly on V, that is, $\Phi_g \in \mathsf{GL}(V)$ for all $g \in G$, where $\mathsf{GL}(V)$ denotes the group of all linear automorphisms of V, then the action is said to be a **representation** of G on V. For example, the adjoint and coadjoint actions of G defined above are representations.

- **Tangent lift of a group action.** Φ induces a natural action on the tangent bundle TM of M by

$$g \cdot v_m := T_m \Phi_g(v_m), \qquad g \in G, \quad v_m \in T_m M.$$

- **Cotangent lift of a group action.** Let $\Phi : G \times M \to M$ be a smooth Lie group action on the manifold M. The map Φ induces a natural action on the cotangent bundle T^*M of M by

$$g \cdot \alpha_m := T^*_{g \cdot m} \Phi_{g^{-1}}(\alpha_m)$$

where $g \in G$ and $\alpha_m \in T^*_m M$.

The **infinitesimal generator** $\xi_M \in \mathfrak{X}(M)$ associated to $\xi \in \mathfrak{g}$ is the vector field on Mdefined by

$$\xi_M(m) := \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp t\xi}(m) = T_e\Phi^m \cdot \xi.$$

The infinitesimal generators are complete vector fields. The flow of ξ_M equals $(t, m) \mapsto \exp t\xi \cdot m$. Moreover, the map $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ is a **Lie algebra antihomomorphism**, that is,

(i) $(a\xi + b\eta)_M = a\xi_M + b\eta_M$,

(ii) $[\xi, \eta]_M = -[\xi_M, \eta_M]$.

If the action is on the right, then $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ is a **Lie algebra homomorphism**.

Let $\mathfrak{g}$ be a Lie algebra and M a smooth manifold. A **(left) right Lie algebra action** of $\mathfrak{g}$ on M is a Lie algebra (anti)homomorphism $\xi \in \mathfrak{g} \longmapsto \xi_M \in \mathfrak{X}(M)$ such that the mapping $(m, \xi) \in M \times \mathfrak{g} \longmapsto \xi_M(m) \in TM$ is smooth.

Given a Lie group action, we will refer to the Lie algebra action induced by its infinitesimal generators as the **associated Lie algebra action**.

Stabilizers and orbits. The **isotropy subgroup** or **stabilizer** of an element m in the manifold M acted upon by the Lie group G is the closed (hence Lie) subgroup

$$G_m := \{g \in G \mid \Phi_g(m) = m\} \subset G$$

whose Lie algebra $\mathfrak{g}_m$ equals

$$\mathfrak{g}_m = \{\xi \in \mathfrak{g} \,|\, \xi_M(m) = 0\}.$$

The **orbit** $\mathcal{O}_m$ of the element $m \in M$ under the group action Φ is the set

$$\mathcal{O}_m \equiv G \cdot m := \{\Phi_g(m) \,|\, g \in G\}.$$

The isotropy subgroups of the elements in a group orbit are related by the expression

$$G_{g\cdot m} = gG_m g^{-1} \text{ for all } g \in G.$$

The notion of orbit allows the introduction of an equivalence relation in the manifold M, namely, two elements $x, y \in M$ are equivalent if and only if they are in the same G–orbit, that is, if there exists an element $g \in G$ such that $\Phi_g(x) = y$. The space of classes with respect to this equivalence relation is usually referred to as the **space of orbits** and, depending on the context, it is denoted by the symbol M/G.

- **Transitive action**: only one orbit, that is, $\mathcal{O}_m = M$

- **Free action**: $G_m = \{e\}$ for all $m \in M$

- **Proper action**: if $\overline{\Phi} : G \times M \to M \times M$ defined by

$$\overline{\Phi}(g, z) := (z, \Phi(g, z))$$

is proper. This is equivalent to: for any two convergent sequences $\{m_n\}$ and $\{g_n \cdot m_n\}$ in M, there exists a convergent subsequence $\{g_{n_k}\}$ in G.

Examples of proper actions: compact group actions, $SE(n)$ acting on $\mathbb{R}^n$, Lie groups acting on themselves by translation.

Fundamental facts about proper Lie group actions

$\Phi : G \times M \to M$ be a proper action of the Lie group G on the manifold M. Then:

(i) The isotropy subgroups G_m are compact.

(ii) The orbit space M/G is a Hausdorff topological space (even when G is not Hausdorff).

(iii) If the action is free, M/G is a smooth manifold, and the canonical projection $\pi : M \to M/G$ defines on M the structure of a smooth left principal G–bundle.

(iv) If all the isotropy subgroups of the elements of M under the G–action are conjugate to a given one H then M/G is a smooth manifold and $\pi : M \to M/G$ defines the structure of a smooth locally trivial fiber bundle with structure group $N(H)/H$ and fiber G/H.

(v) If the manifold M is paracompact then there exists a G-invariant Riemannian metric on it.

(vi) If the manifold M is paracompact then smooth G-invariant functions separate the G-orbits.

Twisted product. Let G be a Lie group and $H \subset G$ a subgroup. Suppose that H acts on the left on the manifold A. The **right twisted action** of H on the product $G \times A$ is defined by

$$(g, a) \cdot h = (gh, h^{-1} \cdot a).$$

This action is free and proper by the freeness and properness of the action on the G–factor. The **twisted product** $G \times_H A$ is defined as the orbit space $(G \times A)/H$ corresponding to the twisted action.

Tube. Let M be a manifold and G a Lie group acting properly on M. Let $m \in M$ and denote $H := G_m$. A **tube** around the orbit $G \cdot m$ is a G-equivariant diffeomorphism

$$\varphi : G \times_H A \longrightarrow U,$$

where U is a G-invariant neighborhood of $G \cdot m$ and A is some manifold on which H acts.

Slice Theorem. G a Lie group acting properly on M at the point $m \in M$, $H := G_m$. There exists a tube

$$\varphi : G \times_H B \longrightarrow U$$

about $G \cdot m$. B is an open H-invariant neighborhood of 0 in a vector space which is H-equivariantly isomorphic to $T_m M / T_m(G \cdot m)$, where the H-representation is given by

$$h \cdot (v + T_m(G \cdot m)) := T_m \Phi_h \cdot v + T_m(G \cdot m).$$

Slice: $S := \varphi([e, B])$ so that $U = G \cdot S$.

Dynamical consequences. $X \in \mathfrak{X}(U)^G$, $U \subset M$ open G-invariant, S slice at $m \in U$. Then there exists

• $X_T \in \mathfrak{X}(G\cdot S)^G$, $X_T(z) = \xi(z)_M(z)$ for $z \in G\cdot S$, where $\xi : G\cdot S \to \mathfrak{g}$ is smooth G-equivariant and $\xi(z) \in \mathrm{Lie}(N(G_z))$ for all $z \in G\cdot S$. The flow T_t of X_T is given by $T_t(z) = \exp t\xi(z)\cdot z$, so X_T is complete.

• $X_N \in \mathfrak{X}(S)^{G_m}$

• If $z = g\cdot s$, for $g \in G$ and $s \in S$, then

$$X(z) = X_T(z) + T_s\Phi_g\left(X_N(s)\right) = T_s\Phi_g\left(X_T(s) + X_N(s)\right)$$

• If N_t is the flow of X_N (on S) then the integral curve of $X \in \mathfrak{X}(U)^G$ through $g \cdot s \in G \cdot S$ is

$$F_t(g \cdot s) = g(t) \cdot N_t(s),$$

where $g(t) \in G$ is the solution of

$$\dot{g}(t) = T_e L_{g(t)} \big(\xi(N_t(s)) \big), \qquad g(0) = g.$$

This is the **tangential-normal** decomposition of a G-invariant vector field (or **Krupa decomposition** in bifurcation theory).

Geometric consequences. **Orbit type**, **fixed point**, and **isotropy type spaces**

$$\begin{aligned} M_{(H)} &= \{z \in M \mid G_z \in (H)\}, \\ M^H &= \{z \in M \mid H \subset G_z\}, \\ M_H &= \{z \in M \mid H = G_z\} \end{aligned}$$

are submanifolds.

$$M_H \quad \text{is open in} \quad M^H.$$

$m \in M$ is **regular** if $\exists U \ni m$ such that $\dim \mathcal{O}_z = \dim \mathcal{O}_m, \forall z \in U$.

Principal Orbit Theorem: M connected. The subset M^{reg} is connected, open, and dense in M. M/G contains only one principal orbit type, which is a connected open and dense subset of it.

The Stratification Theorem: Let M be a smooth manifold and G a Lie group acting properly on it. The connected components of the orbit type manifolds $M_{(H)}$ and their projections onto orbit space $M_{(H)}/G$ constitute a Whitney stratification of M and M/G, respectively. This stratification of M/G is minimal among all Whitney stratifications of M/G.

G-Codostribution Theorem: Let G be a Lie group acting properly on the smooth manifold M and $m \in M$ a point with isotropy subgroup $H := G_m$. Then

$$\left(\left(T_m(G \cdot m)\right)^\circ\right)^H = \left\{\mathbf{d}f(m) \mid f \in C^\infty(M)^G\right\}.$$

SIMPLE EXAMPLES

• **S^1 acting on $\mathbb{R}^2$**

Since S^1 is Abelian we do not distinguish between orbit types and isotropy types, that is, $\mathbb{R}^2_{(H)} = \mathbb{R}^2_H$ for any isotropy group H of this action.

If $\mathbf{x} \neq \mathbf{0}$ then $S^1_{\mathbf{x}} = 1$ and $S^1 \cdot \mathbf{x}$ is the circle centered at the origin of radius $\|\mathbf{x}\|$. The slice is the ray through $\mathbf{0}$ and $\mathbf{x}$. $(\mathbb{R}^2)^{reg} = \mathbb{R}^2 \setminus \{\mathbf{0}\}$, which is open, connected, dense. $\mathbb{R}^2_1 = (\mathbb{R}^2)^{reg}$ and $(\mathbb{R}^2)^{reg}/S^1 =]0, \infty[$.

If $\mathbf{x} = \mathbf{0}$, then $S^1_{\mathbf{0}} = S^1$. The slice is $\mathbb{R}^2$. $\mathbb{R}^2_{\mathbf{0}} = \{\mathbf{0}\}$ and $\mathbb{R}^2_{\mathbf{0}}/S^1 = \{\mathbf{0}\}$.

Finally $\mathbb{R}^2/S^1 = [0, \infty[$.

- SO(3) **acting on** $\mathbb{R}^3$

Since SO(3) is non-Abelian, there is a distinction between orbit and isotropy types.

Since every rotation has an axis, if $\mathbf{x} \neq \mathbf{0}$ the isotropy subgroup $\mathrm{SO}(3)_{\mathbf{x}} = S^1(\mathbf{x})$, the circle representing the rotations with axis $\mathbf{x}$. So $(\mathbb{R}^3)^{reg} = \mathbb{R}^3 \setminus \{\mathbf{0}\}$.

The orbit $\mathrm{SO}(3) \cdot \mathbf{x}$ is the sphere centered at the origin with radius $\|\mathbf{x}\|$. The slice at $\mathbf{x}$ is the ray connecting the origin to $\mathbf{x}$.

$(\mathbb{R}^3)_{S^1(\mathbf{x})}$ is the set of points in $\mathbb{R}^3$ which have the *same* istropy group $S^1(\mathbf{x})$, so it is equal to the line through the origin and $\mathbf{x}$ with the origin eliminated. It is disconnected and *not* SO(3)-invariant.

$(\mathbb{R}^3)_{(S^1(\mathbf{x}))}$ is the set of points in $\mathbb{R}^3$ which have the istropy group $S^1(\mathbf{x})$ conjugate to $S^1(\mathbf{x})$. But any two rotations are conjugate, so $(\mathbb{R}^3)_{(S^1(\mathbf{x}))} = \mathbb{R}^3 \setminus \{\mathbf{0}\}$, which is again equal in this case to $(\mathbb{R}^3)^{reg}$. This is connected, open, dense. $(\mathbb{R}^3)_{(S^1(\mathbf{x}))}/\,\mathsf{SO}(3) =]0,\infty[$.

If $\mathbf{x} = \mathbf{0}$, the slice is $\mathbb{R}^3$, $\mathsf{SO}(3)_{\mathbf{0}} = \mathsf{SO}(3)$, $(\mathbb{R}^3)_{\mathsf{SO}(3)} = (\mathbb{R}^3)_{(\mathsf{SO}(3))} = \{\mathbf{0}\}$, and $(\mathbb{R}^3)_{(\mathsf{SO}(3))} = \{\mathbf{0}\}/\,\mathsf{SO}(3) = \{0\}$.

Finally $\mathbb{R}^3/\,\mathsf{SO}(3) = [0,\infty[$.

- **Semidirect products**

V vector space, G Lie group

$\sigma : G \to \mathsf{GL}(V)$ representation

$\sigma' : \mathfrak{g} \to \mathrm{gl}(V)$ induced Lie algebra representation:

$$\xi \cdot v := \xi_V(v) := \sigma'(\xi)v := \left.\frac{d}{dt}\right|_{t=0} \sigma(\exp t\xi)v$$

$S := G \circledS V$ semidirect product: underlying manifold is $G \times V$, multiplication

$$(g_1, v_1)(g_2, v_2) := (g_1 g_2, v_1 + \sigma(g_1)v_2)$$

for $g_1, g_2 \in G$ and $v_1, v_2 \in V$, identity element is $(e, 0)$ and $(g, v)^{-1} = (g^{-1}, -\sigma(g^{-1})v)$.

Note that V is a normal subgroup of S and that $S/V = G$.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{s} := \mathfrak{g} \circledS V$ be the Lie algebra of S; it is the semidirect product of $\mathfrak{g}$ with V using the representation σ' and its underlying vector space is $\mathfrak{g} \times V$. The Lie bracket on $\mathfrak{s}$ is given by

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \sigma'(\xi_1)v_2 - \sigma'(\xi_2)v_1)$$

for $\xi_1, \xi_2 \in \mathfrak{g}$ and $v_1, v_2 \in V$.

Identify $\mathfrak{s}^*$ with $\mathfrak{g}^* \times V^*$ by using the duality pairing on each factor.

Adjoint action of S on $\mathfrak{s}$:

$$\mathsf{Ad}_{(g,u)}(\xi, v) = \left(\mathsf{Ad}_g \, \xi, \sigma(g)v - \sigma'(\mathsf{Ad}_g \, \xi)u\right),$$

for $(g, u) \in S, (\xi, v) \in \mathfrak{s}$.

Coadjoint action of S on $\mathfrak{s}^*$:

$$\mathsf{Ad}^*_{(g,u)^{-1}}(\nu, a) = \left(\mathsf{Ad}^*_{g^{-1}} \, \nu + (\sigma'_u)^* \sigma_*(g)a, \sigma_*(g)a\right),$$

for $(g, u) \in S$, $(\nu, a) \in \mathfrak{s}^*$, where

$$\sigma_*(g) := \sigma(g^{-1})^* \in \mathsf{GL}(V^*),$$

$\sigma'_u : \mathfrak{g} \to V$ is the linear map given by $\sigma'_u(\xi) := \sigma'(\xi)u$ and $(\sigma'_u)^* : V^* \to \mathfrak{g}^*$ is its dual.

Clasification of orbits is a major problem!

Do the example of the coadjoint action of $SE(3) = SO(3) \circledS \mathbb{R}^3$. In this case:

$\sigma : SO(3) \to \mathrm{GL}(\mathbb{R}^3)$ is usual matrix multiplication on vectors, that is, $\sigma(A)\mathbf{v} := A\mathbf{v}$, for any $A \in SO(3)$ and $\mathbf{v} \in \mathbb{R}^3$.

Dualizing we get $\sigma(A)^*\mathbf{\Gamma} = A^*\mathbf{\Gamma} = A^{-1}\mathbf{\Gamma}$, for any $\mathbf{\Gamma} \in V^* \cong \mathbb{R}^3$.

The induced Lie algebra representation $\sigma' : \mathbb{R}^3 \cong \mathfrak{so}(3) \to \mathfrak{gl}(\mathbb{R}^3)$ is given by $\sigma'(\mathbf{\Omega})\mathbf{v} = \sigma'_{\mathbf{v}}\mathbf{\Omega} = \mathbf{\Omega} \times \mathbf{v}$, for any $\mathbf{\Omega}, \mathbf{v} \in \mathbb{R}^3$.

Therefore, $\left(\sigma'_{\mathbf{v}}\right)^* \mathbf{\Gamma} = \mathbf{v} \times \mathbf{\Gamma}$ and $\sigma'(\mathbf{\Omega})^* \mathbf{\Gamma} = \mathbf{\Gamma} \times \mathbf{\Omega}$, for any $\mathbf{v} \in V \cong \mathbb{R}^3$, $\mathbf{\Omega} \in \mathbb{R}^3 \cong \mathfrak{so}(3)$, and $\mathbf{\Gamma} \in V^* \cong \mathbb{R}^3$.

We have $\mathrm{ad}^*_{\mathbf{\Omega}} \mathbf{\Pi} = \mathbf{\Pi} \times \mathbf{\Omega}$

So all formulas in this case become:

$$(\mathbf{A}, \mathbf{a})(\mathbf{B}, \mathbf{b}) = (\mathbf{AB}, \mathbf{Ab} + \mathbf{a})$$

$$(\mathbf{A}, \mathbf{a})^{-1} = (\mathbf{A}^{-1}, -\mathbf{A}^{-1}\mathbf{a})$$

$$[(\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y}')] = (\mathbf{x}\times\mathbf{x}', \mathbf{x}\times\mathbf{y}' - \mathbf{x}'\times\mathbf{y})$$

$$\mathsf{Ad}_{(\mathbf{A},\mathbf{a})}(\mathbf{x},\mathbf{y}) = (\mathbf{A}\mathbf{x}, \mathbf{A}\mathbf{y} - \mathbf{A}\mathbf{x}\times\mathbf{a})$$

$$\mathsf{Ad}^*_{(\mathbf{A},\mathbf{a})^{-1}}(\mathbf{u},\mathbf{v}) = (\mathbf{A}\mathbf{u} + \mathbf{a}\times\mathbf{A}\mathbf{v}, \mathbf{A}\mathbf{v})$$

Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3\}$ be an orthonormal basis of $\mathfrak{se}(3) = \mathbb{R}^3\times\mathbb{R}^3$ such that $\mathbf{e}_i = \mathbf{f}_i$ for $i = 1,2,3$. The dual basis of $\mathfrak{se}(3)^*$ using the dot product is again $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3\}$. Let $\mathbf{e}\in\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\mathbf{f}\in\{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3\}$ be arbitrary. What are the coadjoint orbits?

$\mathsf{SE}(3)\cdot(\mathbf{0},\mathbf{0}) = (\mathbf{0},\mathbf{0})$. Since $\mathsf{SE}(3)_{(\mathbf{0},\mathbf{0})} = \mathsf{SE}(3)$ is not compact, *the coadjoint action is not proper*.

The orbit through $(\mathbf{e}, \mathbf{0})$, $\mathbf{e} \neq \mathbf{0}$, is

$$\mathrm{SE}(3) \cdot (\mathbf{e}, \mathbf{0}) = \{ (\mathbf{A}\mathbf{e}, \mathbf{0}) \mid \mathbf{A} \in \mathrm{SO}(3) \} = S^2_{\|\mathbf{e}\|} \times \{\mathbf{0}\},$$

the two-sphere of radius $\|\mathbf{e}\|$.

The orbit through $(\mathbf{0}, \mathbf{f})$, $\mathbf{f} \neq \mathbf{0}$, is

$$\begin{aligned} \mathrm{SE}(3) \cdot (\mathbf{0}, \mathbf{f}) &= \{ (\mathbf{a} \times \mathbf{A}\mathbf{f}, \mathbf{A}\mathbf{f}) \mid \mathbf{A} \in \mathrm{SO}(3), \, \mathbf{a} \in \mathbb{R}^3 \} \\ &= \{ (\mathbf{u}, \mathbf{A}\mathbf{f}) \mid \mathbf{A} \in \mathrm{SO}(3), \, \mathbf{u} \perp \mathbf{A}\mathbf{f} \} = TS^2_{\|\mathbf{f}\|}, \end{aligned}$$

the tangent bundle of the two-sphere of radius $\|\mathbf{f}\|$; note that the vector part is the first component. We can think of it also as $T^*S^2_{\|\mathbf{f}\|}$.

The orbit through $(\mathbf{e}, \mathbf{f})$, where $\mathbf{e} \neq \mathbf{0}, \mathbf{f} \neq \mathbf{0}$, equals

$$\mathsf{SE}(3) \cdot (\mathbf{e}, \mathbf{f}) = \{ (\mathbf{Ae} + \mathbf{a} \times \mathbf{Af}, \mathbf{Af}) \mid \mathbf{A} \in \mathsf{SO}(3), \, \mathbf{a} \in \mathbb{R}^3 \}.$$

To get a better description of this orbit, consider the smooth map

$$\varphi : (\mathbf{A}, \mathbf{a}) \in \mathsf{SE}(3) \mapsto \left(\mathbf{Ae} + \mathbf{a} \times \mathbf{Af} - \frac{\mathbf{e} \cdot \mathbf{f}}{\|\mathbf{f}\|^2} \mathbf{Af}, \mathbf{Af} \right) \in TS^2_{\|\mathbf{f}\|},$$

which is right invariant under the isotropy group

$$\mathsf{SE}(3)_{(\mathbf{e},\mathbf{f})} = \{ (\mathbf{B}, \mathbf{b}) \mid \mathbf{Be} + \mathbf{b} \times \mathbf{f} = \mathbf{e}, \ \mathbf{Bf} = \mathbf{f} \}$$

and induces hence a diffeomorphism $\bar{\varphi} : \mathsf{SE}(3) / \mathsf{SE}(3)_{(\mathbf{e},\mathbf{f})} \to TS^2_{\|\mathbf{f}\|}$.

The orbit through $(\mathbf{e},\mathbf{f})$ is diffeomorphic to $\mathrm{SE}(3)/\,\mathrm{SE}(3)_{(\mathbf{e},\mathbf{f})}$ by the diffeomorphism

$$(\mathbf{A},\mathbf{a}) \mapsto \mathrm{Ad}^*_{(\mathbf{A},\mathbf{a})^{-1}}(\mathbf{e},\mathbf{f}).$$

Composing these two maps and identifying TS^2 and T^*S^2 by the natural Riemannian metric on S^2, we get the diffeomorphism $\Phi : \mathrm{SE}(3)\cdot(\mathbf{e},\mathbf{f}) \to T^*S^2_{\|\mathbf{f}\|}$ given by

$$\Phi(\mathrm{Ad}^*_{(\mathbf{A},\mathbf{a})^{-1}}(\mathbf{e},\mathbf{f})) = \left(\mathbf{A}\mathbf{e} + \mathbf{a}\times\mathbf{A}\mathbf{f} - \frac{\mathbf{e}\cdot\mathbf{f}}{\|\mathbf{f}\|^2}\mathbf{A}\mathbf{f}, \mathbf{A}\mathbf{f}\right).$$

Thus this orbit is also diffeomorphic to $T^*S^2_{\|\mathbf{f}\|}$.

• SE(3) **acting on** $\mathbb{R}^3$

This action is proper: $(\mathbf{A}, \mathbf{a}) \cdot \mathbf{u} := \mathbf{Au} + \mathbf{a}$. It is not a representation. The orbit through the origin is $\mathbb{R}^3$, $SE(3)_{\mathbf{0}} = SO(3)$.

This action is transitive: given $\mathbf{u} \in \mathbb{R}^3$ we have $(\mathbf{I}, \mathbf{0}) \cdot \mathbf{u} = \mathbf{u}$. So there is only one single orbit which is $\mathbb{R}^3$.

EXAMPLE

- Consider $\mathbb{R}^6$ with the bracket

$$\{f, g\} = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right)$$

- S^1-action given by

$$\begin{array}{rccc} \Phi : & S^1 \times \mathbb{R}^6 & \longrightarrow & \mathbb{R}^6 \\ & (e^{i\phi}, (\mathbf{x}, \mathbf{y})) & \longmapsto & (R_\phi \mathbf{x}, R_\phi \mathbf{y}) \end{array}$$

- Hamiltonian of the spherical pendulum

$$h = \frac{1}{2} \langle y, y \rangle + \langle x, e_3 \rangle$$

- Impose constraint $\langle x, x \rangle = 1$

- Angular momentum: $\mathbf{J}(\mathbf{x}, \mathbf{y}) = x_1 y_2 - x_2 y_1$.

Hilbert-Weyl Theorem: $H \to \text{Aut}(V)$ representation, H compact Lie group. Then the algebra $\mathcal{P}(V)^H$ of H-invariant polynomials on V is finitely generated, i.e., $\forall P \in \mathcal{P}(V)^H, \exists k \in \mathbb{N}, \pi_1, \ldots, \pi_k \in \mathcal{P}(V)^H, \widehat{P} \in \mathbb{R}[X_1, \ldots, X_k]$ s.t. $P = \widehat{P} \circ (\pi_1, \ldots, \pi_k)$. Minimal set is a **Hilbert basis**.

Hilbert basis of the algebra of S^1-invariant polynomials on $\mathbb{R}^6$ is given by

$$
\begin{array}{lll}
\sigma_1 = x_3 & \sigma_3 = y_1^2 + y_2^2 + y_3^2 & \sigma_5 = x_1^2 + x_2^2 \\
\sigma_2 = y_3 & \sigma_4 = x_1 y_1 + x_2 y_2 & \sigma_6 = x_1 y_2 - x_2 y_1.
\end{array}
$$

Semialgebraic relations

$$\sigma_4^2 + \sigma_6^2 = \sigma_5(\sigma_3 - \sigma_2^2), \qquad \sigma_3 \geq 0, \qquad \sigma_5 \geq 0.$$

Hilbert map $\pi : v \in V \mapsto (\pi_1(v), \ldots, \pi_k(v)) \in \mathbb{R}^k$ separates H-orbits. So $V/H \cong \text{range}(\pi)$.

Schwarz Theorem: The map $f \in C^\infty(\mathbb{R}^k) \mapsto f \circ (\pi_1, \ldots \pi_k) \in C^\infty(V)^H$ is surjective.

Mather Theorem: The quotient presheaf of smooth functions on V/H is isomorphic to the presheaf of Whitney smooth functions on $\pi(V)$ induced by the sheaf of smooth functions on $\mathbb{R}^k$.

Tarski-Seidenberg Theorem: Since π is a polynomial map, $\text{range}(\pi) \subset \mathbb{R}^k$ is semi-algebraic.

Theorem: Every semi-algebraic set admits a canonical Whitney stratification into a finite number of semi-algebraic subsets.

Bierstone Theorem: This canonical stratification of $\pi(V)$ coincides with the stratification of V/H into orbit type manifolds.

These theorems can be used to explicitly describe quotient spaces of representations as semi-algebraic subsets of a (high dimensional) Euclidean space.

Return to our concrete case of the spherical pendulum.

The Hilbert map is given by

$$\begin{array}{rccc} \sigma : & T\mathbb{R}^3 & \longrightarrow & \mathbb{R}^6 \\ & (\mathbf{x}, \mathbf{y}) & \longmapsto & (\sigma_1(\mathbf{x}, \mathbf{y}), \ldots, \sigma_6(\mathbf{x}, \mathbf{y})). \end{array}$$

The S^1-orbit space $T\mathbb{R}^3/S^1$ can be identified with the semialgebraic variety $\sigma(T\mathbb{R}^3) \subset \mathbb{R}^6$, defined by these relations.

TS^2 is a submanifold of $\mathbb{R}^6$ given by

$$TS^2 = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1,\ \langle \mathbf{x}, \mathbf{y} \rangle = 0\}.$$

TS^2 is S^1-invariant.

TS^2/S^1 can be thought of the semialgebraic variety $\sigma(TS^2)$ defined by the previous relations and

$$\sigma_5 + \sigma_1^2 = 1 \qquad \sigma_4 + \sigma_1\sigma_2 = 0,$$

which allow us to solve for σ_4 and σ_5, yielding

$$\begin{aligned} TS^2/S^1 = \sigma(TS^2) = \{(\sigma_1,\, \sigma_2,\, \sigma_3,\, \sigma_6) \in \mathbb{R}^4 \mid \\ \sigma_1^2\sigma_2^2 + \sigma_6^2 = (1-\sigma_1^2)(\sigma_3 - \sigma_2^2), \\ |\sigma_1| \leq 1,\, \sigma_3 \geq 0\}. \end{aligned}$$

The Poisson bracket is

$\{\cdot,\cdot\}^{TS^2/S^1}$	σ_1	σ_2	σ_3	σ_6
σ_1	0	$1-\sigma_1^2$	$2\sigma_2$	0
σ_2	$-(1-\sigma_1^2)$	0	$-2\sigma_1\sigma_3$	0
σ_3	$-2\sigma_2$	$2\sigma_1\sigma_3$	0	0
σ_6	0	0	0	0

The reduced Hamiltonian is

$$H = \frac{1}{2}\sigma_3 + \sigma_1$$

If $\mu \neq 0$ then $(TS^2)_\mu := \mathbf{J}^{-1}(\mu)/S^1$ appears as the graph of the smooth function

$$\sigma_3 = \frac{\sigma_2^2 + \mu^2}{1 - \sigma_1^2}, \qquad |\sigma_1| < 1.$$

The case $\mu = 0$ is singular and $(TS^2)_0 := \mathbf{J}^{-1}(0)/S^1$ is not a smooth manifold.

ABSTRACT SYMMETRY REDUCTION

The case of general vector fields

M manifold

$G \times M \to M$ smooth proper Lie group action

$X \in \mathfrak{X}(M)^G$, G-equivariant vector field

F_t flow of $X \in \mathfrak{X}(M)^G$

Law of conservation of isotropy:

$M_H := \{m \in M \mid G_m = H\}$, the H-**isotropy type submanifold**, is preserved by F_t.

M_H is, in general, not closed in M.

Properness of the action implies:

- G_m is compact
- the (connected components of) M_H are embedded submanifolds of M

$N(H)/H$ (where $N(H)$ denotes the normalizer of H in G) acts freely and properly on M_H.

$\pi_H : M_H \to M_H/(N(H)/H)$ projection

$i_H : M_H \hookrightarrow M$ inclusion

X induces a unique H-**isotropy type reduced vector field** X^H on $M_H/(N(H)/H)$ by

$$X^H \circ \pi_H = T\pi_H \circ X \circ i_H,$$

whose flow F_t^H is given by

$$F_t^H \circ \pi_H = \pi_H \circ F_t \circ i_H.$$

If G is compact and the action is linear, then the construction of $M_H/(N(H)/H)$ can be implemented in a very explicit and convenient manner by using the invariant polynomials of the action and the theorems of Hilbert and Schwarz-Mather.

The Hamiltonian case

(M, ω) Poisson manifold, G connected Lie group with Lie algebra $\mathfrak{g}$, $G \times M \to M$ free proper symplectic action

$\mathbf{J} : M \to \mathfrak{g}^*$ **momentum map** if $X_{\mathbf{J}^\xi} = \xi_M$, where $\mathbf{J}^\xi := \langle \mathbf{J}, \xi \rangle$ and ξ_M is the infinitesimal generator given by $\xi \in \mathfrak{g}$

$\mathbf{J} : M \to \mathfrak{g}^*$ **(infinitesimally) equivariant** if $\mathbf{J}(g \cdot m) = \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(m)$, $\forall g \in G$ $\left(T_m \mathbf{J} \left(\xi_M(m) \right) = -\,\mathrm{ad}^*_\xi \mathbf{J}(m) \iff \mathbf{J}^{[\xi, \eta]} = \left\{ \mathbf{J}^\xi, \mathbf{J}^\eta \right\} \right)$.

Proof Take the derivative on M of the defining relation $\mathbf{J}^{\xi} := \langle \mathbf{J}, \xi \rangle$. Get: $\mathbf{dJ}^{\xi}(m)(v_m) = \langle T_m\mathbf{J}(v_m), \xi \rangle$. Hence

$$\left\{ \mathbf{J}^{\xi}, \mathbf{J}^{\eta} \right\}(m) = X_{\mathbf{J}^{\eta}}\left[\mathbf{J}^{\xi} \right](m) = \mathbf{dJ}^{\xi}(m)\left(X_{\mathbf{J}^{\eta}}(m) \right)$$
$$= \left\langle T_m\mathbf{J}\left(X_{\mathbf{J}^{\eta}}(m) \right), \xi \right\rangle = \left\langle T_m\mathbf{J}\left(\eta_M(m) \right), \xi \right\rangle .$$

On the other hand,

$$\mathbf{J}^{[\xi,\eta]}(m) = \langle \mathbf{J}(m), [\xi, \eta] \rangle = - \left\langle \mathbf{J}(m), \mathrm{ad}_{\eta}\, \xi \right\rangle$$
$$= - \left\langle \mathrm{ad}_{\eta}^{*}\, \mathbf{J}(m), \xi \right\rangle .$$

Noether's Theorem: The fibers of $\mathbf{J}$ are preserved by the Hamiltonian flows associated to G-invariant Hamiltonians. Equivalently, $\mathbf{J}$ is conserved along the flow of any G-invariant Hamiltonian.

Proof Let $h \in C^\infty(M)$ be G-invariant, so $h \circ \Phi_g = h$ for any $g \in G$. Take the derivative of this relation at $g = e$ and get $\pounds_{\xi_M} h = 0$. But $\xi_M = X_{\mathbf{J}^\xi}$ so we get $\{\mathbf{J}^\xi, h\} = \left\langle \mathbf{d}h, X_{\mathbf{J}^\xi} \right\rangle = \pounds_{\xi_M} h = 0$, which shows that $\mathbf{J}^\xi \in C^\infty(M)$ is constant on the flow of X_h for any $\xi \in \mathfrak{g}$, that is $\mathbf{J}$ is conserved. $\square$

Example: lifted actions on cotangent bundles. $\Phi : G \times Q \to Q$ Lie group action, $g \cdot q := \Phi(g, q)$. Its lift to the cotangent bundle T^*Q is

$$g \cdot \alpha_q := \Psi_g \alpha_q := T^*_{g \cdot q} \Phi_{g^{-1}}(\alpha_q).$$

Ψ admits the following equivariant momentum map:

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle, \quad \forall \alpha_q \in T^*Q, \quad \forall \xi \in \mathfrak{g}.$$

Very important so we will give two complete proofs.

Proof 1 Recall that the cotangent lift of a diffeomorphism preserves the canonical one-form Θ on T^*Q. Hence $\Psi^*_{\exp t\xi}\Theta = \Theta$. Take $\left.\frac{d}{dt}\right|_{t=0}$ of this:

$$0 = \pounds_{\xi_{T^*Q}}\Theta = \mathbf{i}_{\xi_{T^*Q}}\mathbf{d}\Theta + \mathbf{d}\mathbf{i}_{\xi_{T^*Q}}\Theta = -\mathbf{i}_{\xi_{T^*Q}}\Omega + \mathbf{d}\left\langle \Theta, \xi_{T^*Q} \right\rangle$$

which shows that a momentum map exists and is equal to $\mathbf{J}^\xi = \left\langle \Theta, \xi_{T^*Q} \right\rangle$. However, $\forall \alpha_q \in T^*Q$, we have

$$\mathbf{J}^\xi(\alpha_q) = \left\langle \Theta(\alpha_q), \xi_{T^*Q}(\alpha_q) \right\rangle = \left\langle \alpha_q, T_{\alpha_q}\pi_Q \left(\xi_{T^*Q}(\alpha_q) \right) \right\rangle .$$

But

$$\begin{aligned} T_{\alpha_q}\pi_Q \left(\xi_{T^*Q}(\alpha_q) \right) &= T_{\alpha_q}\pi_Q \left(\left.\frac{d}{dt}\right|_{t=0} \Psi_{\exp t\xi}(\alpha_q) \right) \\ &= \left.\frac{d}{dt}\right|_{t=0} \left(\pi_Q \circ \Psi_{\exp t\xi} \right)(\alpha_q) = \left.\frac{d}{dt}\right|_{t=0} \left(\Phi_{\exp t\xi} \circ \pi_Q \right)(\alpha_q) \\ &= \xi_Q(q), \end{aligned}$$

which proves the formula.

We prove G-equivariance. Let $g \in G$, $\xi \in \mathfrak{g}$, $\alpha_q \in T^*Q$.

$$
\begin{aligned}
\langle \mathbf{J}(g \cdot \alpha_q), \xi \rangle &= \left\langle g \cdot \alpha_q, \xi_Q(g \cdot q) \right\rangle \\
&= \left\langle \alpha_q, \left(T_{g \cdot q} \Phi_g^{-1} \circ \xi_Q \circ \Phi_g \right)(q) \right\rangle = \left\langle \alpha_q, \left(\mathrm{Ad}_{g^{-1}} \xi \right)_Q (q) \right\rangle \\
&= \left\langle \mathbf{J}(\alpha_q), \mathrm{Ad}_{g^{-1}} \xi \right\rangle = \left\langle \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(\alpha_q), \xi \right\rangle . \qquad \square
\end{aligned}
$$

Proof 2 Define the **momentum function** of $X \in \mathfrak{X}(Q)$

$$\mathcal{P} : \mathfrak{X}(Q) \to C^\infty(T^*Q) \quad \text{by} \quad \mathcal{P}(X)(\alpha_q) := \langle \alpha_q, X(q) \rangle$$

for any $\alpha_q \in T_q^*Q$. In coordinates $\mathcal{P}(q^i, p_i) = X^j(p_i)p_j$.

$\mathcal{L}(T^*Q)$ is the space of **smooth functions linear on the fibers**. In coordinates $F \in \mathcal{L}(T^*Q) \iff F(q^i, p_i) = X^j(q^i)p_j$ for some functions X^j. If $H(q^i, p_i) = Y^j(q^i)p_j$,

$$\begin{aligned}\{F, H\}(q^i, p_i) &= \frac{\partial F}{\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q^j}\frac{\partial F}{\partial p_j} \\ &= \frac{\partial X^i}{\partial q^j} p_i Y^k \delta^j_k - \frac{\partial Y^i}{\partial q^j} p_i X^k \delta^j_k \\ &= \left(\frac{\partial X^i}{\partial q^j} p_i Y^j - \frac{\partial Y^i}{\partial q^j} p_i X^j \right) p_i\end{aligned}$$

so $\mathcal{L}(T^*Q)$ is a Lie subalgebra of $C^\infty(T^*Q)$.

Momentum Commutator Lemma: The Lie algebras

(i) $(\mathfrak{X}(Q), [\cdot, \cdot])$ of vector fields on Q

(ii) Hamiltonian vector fields X_F on T^*Q with $F \in \mathcal{L}(T^*Q)$

are isomorphic. Each of these Lie algebras is anti-isomorphic to $(\mathcal{L}(T^*Q), \{\cdot, \cdot\})$. In particular, we have

$$\{\mathcal{P}(X), \mathcal{P}(Y)\} = -\mathcal{P}([X, Y]).$$

Proof $\mathcal{P} : \mathfrak{X}(Q) \to \mathcal{L}(T^*Q)$ is linear and satisfies the relation above because

$$[X, Y]^i = \frac{\partial Y^i}{\partial q^j} X^j - \frac{\partial X^i}{\partial q^j} Y^j$$

implies

$$-\mathcal{P}([X,Y]) = \left(\frac{\partial X^i}{\partial q^j} p_i Y^j - \frac{\partial Y^i}{\partial q^j} p_i X^j\right) p_i = \{\mathcal{P}(X), \mathcal{P}(Y)\}$$

as we saw above. So, $\mathcal{P}$ is a Lie algebra anti-homomorphism.

$\mathcal{P}(X) = 0 \iff \mathcal{P}(X)(\alpha_q) := \langle \alpha_q, X(q) \rangle, \forall \alpha_q \in T^*Q \iff$ $X(q) = 0, \forall q \in Q$, so $\mathcal{P}$ is injective.

For each $F \in \mathcal{L}(T^*Q)$, define $X(F) \in \mathfrak{X}(Q)$ by

$$\langle \alpha_q, X(F)(q) \rangle := F(\alpha_q).$$

Then $\mathcal{P}(X(F)) = F$, so $\mathcal{P}$ is also surjective.

We know that $F \mapsto X_F$ is a Lie algebra anti-homomorphism (by the Jacobi identity for $\{\cdot,\cdot\}$) from $(\mathcal{L}(T^*Q),\{\cdot,\cdot\})$ to $(\{X_F \mid F \in \mathcal{L}(T^*Q)\},[\cdot,\cdot])$. This map is surjective by definition. Moreover, if $X_F = 0$ then F is constant on T^*Q, hence equal to zero becuase F is linear on the fibers. $\square$

If $X \in \mathfrak{X}(Q)$ has flow φ_t, then the flow of $X_{\mathcal{P}(X)}$ on T^*Q is $T^*\varphi_{-t}$. Call $X' := X_{\mathcal{P}(X)}$ the **cotangent lift** of X.

Proof $\pi_Q : T^*Q \to Q$ cotangent bundle projection. Differentiate $\pi_Q \circ T^*\varphi_{-t} = \varphi_t \circ \pi_Q$ at $t = 0$ and get

$$T\pi_Q \circ Y = X \circ \pi_Q, \quad \text{where} \quad Y(\alpha_q) := \left.\frac{d}{dt}\right|_{t=0} T^*\varphi_{-t}(\alpha_q)$$

So, $T^*\varphi_{-t}$ is the flow of Y, by construction. Since $T^*\varphi_{-t}$ preserves the canonical one-form $\Theta \in \Omega^1(T^*Q)$, it follows that $\pounds_Y \Theta = 0$, hence

$$\mathbf{i}_Y \Omega = -\mathbf{i}_Y \mathbf{d}\Theta = \mathbf{d}\mathbf{i}_Y \Theta - \pounds_Y \Theta = \mathbf{d}\mathbf{i}_Y \Theta$$

By definition of Θ, we have

$$\begin{aligned} \mathbf{i}_Y \Theta(\alpha_q) &= \langle \Theta(\alpha_q), Y(\alpha_q) \rangle = \left\langle \alpha_q, T_{\alpha_q}\pi_Q(Y(\alpha_q)) \right\rangle \\ &= \langle \alpha_q, X(q) \rangle = \mathcal{P}(X)(\alpha_q) \iff \mathbf{i}_Y \Theta = \mathcal{P}(X), \end{aligned}$$

that is, $\mathbf{i}_Y \Omega = \mathbf{d}\mathcal{P}(X) \Longleftrightarrow Y = X_{\mathcal{P}(X)}$. $\square$

Note:

$$\left[X_{\mathcal{P}(X)}, X_{\mathcal{P}(Y)}\right] = -X_{\{\mathcal{P}(X), \mathcal{P}(Y)\}} = -X_{-\mathcal{P}([X,Y])} = X_{\mathcal{P}([X,Y])}$$

$\mathfrak{g}$ acts on the left on Q, so it acts on T^*Q by $\xi_{T^*Q} := X_{\mathcal{P}(\xi_Q)}$. This $\mathfrak{g}$-action on T^*Q is Hamiltonian with infinitesimally equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$ given by

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \left\langle \alpha_q, \xi_Q(q) \right\rangle = \mathcal{P}(\xi_Q)(\alpha_q)$$

If G, with Lie algebra $\mathfrak{g}$, acts on Q and hence on T^*Q by cotangent lift, then $\mathbf{J}$ is equivariant.

In coordinates, $\xi^i_Q(q^j) = \xi^a A^i_a(q^j) \Rightarrow J_a \xi^a = p_i \xi^i_Q = p_i A^i_a \xi^a$, i.e.,

$$J_a(q^j, p_j) = p_i A^i_a(q^j)$$

Proof For Lie group actions, the theorem follows directly from the previous one, because the infinitesimal generator is given by $\xi_{T^*Q} := X_{\mathcal{P}(\xi_Q)}$, so the momentum map exists and is given by $\mathbf{J}^\xi = \mathcal{P}(\xi_Q)$ for all $\xi \in \mathfrak{g}$.

For Lie algebra actions we need to check first that the cotangent lift gives a canonical action. So, for $\xi, \eta \in \mathfrak{g}$,

$$\begin{aligned}
\xi_{T^*Q}[\{F,H\}] &= X_{\mathcal{P}(\xi_Q)}[\{F,H\}] \\
&= \Big\{X_{\mathcal{P}(\xi_Q)}[F], H\Big\} + \Big\{F, X_{\mathcal{P}(\xi_Q)}[H]\Big\} \\
&= \big\{\xi_{T^*Q}[F], H\big\} + \big\{F, \xi_{T^*Q}[H]\big\}
\end{aligned}$$

Done!

Remember that the momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is given by $\mathbf{J}^\xi = \mathcal{P}(\xi_Q)$ for any $\xi \in \mathfrak{g}$.

Recall the formula $[\xi, \eta]_Q = -[\xi_Q, \eta_Q]$. Then

$$\begin{aligned}\mathbf{J}^{[\xi,\eta]} &= \mathcal{P}([\xi, \eta]_Q) = -\mathcal{P}([\xi_Q, \eta_Q]) = \left\{\mathcal{P}(\xi_Q), \mathcal{P}(\eta_Q)\right\} \\ &= \left\{\mathbf{J}^\xi, \mathbf{J}^\eta\right\},\end{aligned}$$

so $\mathbf{J}$ is infinitesimally equivariant.

Now assume that G has Lie algebra $\mathfrak{g}$ and that G acts on Q and hence on T^*Q by cotangent lift. Remember: $g \cdot \alpha_q := T^*_{g \cdot q} \Phi_{g^{-1}} \alpha_q$.

We prove G-equivariance. Let $g \in G$, $\xi \in \mathfrak{g}$, $\alpha_q \in T^*Q$.

$$\begin{aligned}
\langle \mathbf{J}(g \cdot \alpha_q), \xi \rangle &= \left\langle g \cdot \alpha_q, \xi_Q(g \cdot q) \right\rangle \\
&= \left\langle \alpha_q, \left(T_{g \cdot q} \Phi_g^{-1} \circ \xi_Q \circ \Phi_g \right)(q) \right\rangle \\
&= \left\langle \alpha_q, \left(\mathrm{Ad}_{g^{-1}} \xi \right)_Q (q) \right\rangle \\
&= \left\langle \mathbf{J}(\alpha_q), \mathrm{Ad}_{g^{-1}} \xi \right\rangle \\
&= \left\langle \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(\alpha_q), \xi \right\rangle . \qquad \square
\end{aligned}$$

If $\mathbf{J} : M \to \mathfrak{g}^*$ is an infinitesimally equivariant momentum map for a left Hamiltonian action of $\mathfrak{g}$ on a Poisson manifold M, then $\mathbf{J}$ is a Poisson map:

$$\mathbf{J}^*\{F_1, F_2\}_+ = \{\mathbf{J}^* F_1, \mathbf{J}^* F_2\}, \ \ \forall F_1, F_2 \in C^\infty(\mathfrak{g}^*).$$

Proof Infinitesimal equivariance $\Leftrightarrow \{\mathbf{J}^\xi, \mathbf{J}^\eta\} = \mathbf{J}^{[\xi,\eta]}$. Let $m \in M$, $\xi = \delta F_1/\delta\mu$, $\eta = \delta F_2/\delta\mu$, $\mu := \mathbf{J}(m) \in \mathfrak{g}^*$. Then

$$\begin{aligned}\mathbf{J}^*\{F_1, F_2\}_+(m) &= \left\langle \mu, \left[\frac{\delta F_1}{\delta\mu}, \frac{\delta F_2}{\delta\mu}\right]\right\rangle = \langle \mu, [\xi, \eta]\rangle \\ &= \mathbf{J}^{[\xi,\eta]}(m) = \{\mathbf{J}^\xi, \mathbf{J}^\eta\}(m).\end{aligned}$$

But for any $m \in M$ an $v_m \in T_mM$, we have

$$\begin{aligned}\mathbf{d}(F_1 \circ \mathbf{J})(m)(v_m) &= \mathbf{d}F_1(\mu)\left(T_m\mathbf{J}(v_m)\right) \\ &= \left\langle T_m\mathbf{J}(v_m), \frac{\delta F_1}{\delta\mu}\right\rangle = \mathbf{d}\mathbf{J}^\xi(m)(v_m)\end{aligned}$$

i.e., $F_1 \circ \mathbf{J}$ and $\mathbf{J}^\xi$ have equal m-derivatives. The Poisson bracket depends only on the point values of the first derivatives and hence

$$\{F_1 \circ \mathbf{J}, F_2 \circ \mathbf{J}\}(m) = \{\mathbf{J}^\xi, \mathbf{J}^\eta\}(m). \qquad \square$$

Special case: $M = T^*G$, G-action on T^*G is the lift of left translation. We get: $\{F_1, F_2\}_+ \circ \mathbf{J}_L = \{F_1 \circ \mathbf{J}_L, F_2 \circ \mathbf{J}_L\}$. Restrict this relation to $\mathfrak{g}^*$ and get $\{F_1, F_2\}_+(\mu) = \{F_1 \circ \mathbf{J}_L, F_2 \circ \mathbf{J}_L\}(\mu)$. But $(F_i \circ \mathbf{J}_L)(\alpha_g) = F_i(T_e^* R_g \alpha_g) =: (F_i)_R(\alpha_g)$, where $(F_i)_R : T^*G \to \mathfrak{g}^*$ is the right invariant extension of F_i to T^*G. So we get

$$\{F_1, F_2\}_+(\mu) = \{(F_1)_R, (F_2)_R\}(\mu).$$

Identifying the set of functions on $\mathfrak{g}^*$ with the set of right(left)-invariant functions on T^*G endows $\mathfrak{g}^*$ with the $\pm$Lie-Poisson structure.

This is an *a posteriori* proof, i.e., one needs to already know the formula for the Lie-Poisson bracket.

Example: linear momentum. Take the phase space of the N−particle system, that is, $T^*\mathbb{R}^{3N}$. The additive group $\mathbb{R}^3$ acts on it by

$$\mathbf{v}\cdot(\mathbf{q}_i,\ \mathbf{p}^i) = (\mathbf{q}_i+\mathbf{v},\ \mathbf{p}^i) \ \Rightarrow\ \xi_{\mathbb{R}^3}(\mathbf{q}_i) = (\mathbf{q}_1, \ldots, \mathbf{q}_N; \xi, \ldots, \xi).$$

$$\begin{array}{rrcl} \mathbf{J}: & T^*\mathbb{R}^{3N} & \longrightarrow & \mathsf{Lie}(\mathbb{R}^3) \simeq \mathbb{R}^3 \\ & (\mathbf{q}_i,\, \mathbf{p}^i) & \longmapsto & \sum_{i=1}^N \mathbf{p}^i \end{array}$$

which is the classical **linear momentum**.

Indeed, by the general formula ofcotangent lifted actions, we have

$$\left\langle \mathbf{J}(\mathbf{q}_i, \mathbf{p}^i), \xi \right\rangle = \sum_{i=1}^N \mathbf{p}^i \cdot \xi.$$

Example: angular momentum. Let SO(3) act on $\mathbb{R}^3$ and then, by lift, on $T^*\mathbb{R}^3$, that is, $A \cdot (\mathbf{q}, \mathbf{p}) = (A\mathbf{q}, A\mathbf{p})$.

$$\begin{array}{rrcl} \mathbf{J}: & T^*\mathbb{R}^3 & \longrightarrow & \mathfrak{so}(3)^* \simeq \mathbb{R}^3 \\ & (\mathbf{q}, \mathbf{p}) & \longmapsto & \mathbf{q} \times \mathbf{p}. \end{array}$$

which is the classical **angular momentum**.

Let's do it using the formula for cotangent lifted actions.

If $\xi \in \mathbb{R}^3$, $\widehat{\xi}\mathbf{v} := \xi \times \mathbf{v}$, for any $\mathbf{v} \in \mathbb{R}^3$, $\widehat{\xi} \in \mathfrak{so}(3)$, then

$$\xi_{\mathbb{R}^3}(\mathbf{v}) = \left.\frac{d}{dt}\right|_{t=0} e^{t\widehat{\xi}}\mathbf{v} = \widehat{\xi}\mathbf{v} = \xi \times \mathbf{v}$$

so that

$$\langle \mathbf{J}(\mathbf{q}, \mathbf{p}), \xi \rangle = \mathbf{p} \cdot \xi_{\mathbb{R}^3}(\mathbf{q}) = \mathbf{p} \cdot (\xi \times \mathbf{q}) = (\mathbf{q} \times \mathbf{p}) \cdot \xi$$

which shows that

$$\mathbf{J}(\mathbf{q}, \mathbf{p}) = \mathbf{q} \times \mathbf{p}$$

Example: Momentum map of the cotangent lifted left and right translations. Let G act on itself on the left: $L_g(h) := gh$. The infinitesimal generator of $\xi \in \mathfrak{g}$ is

$$\xi_G^L(h) := \frac{d}{dt}\bigg|_{t=0} L_{\exp t\xi}(h) = \frac{d}{dt}\bigg|_{t=0} R_h(\exp t\xi) = T_e R_h \xi$$

The infinitesimal generator of left translation is given by the tangent map of right translation: $\xi_G^L(h) = T_e R_h \xi$.

The momentum map of the cotangent lift of left translation $\mathbf{J}_L : T^*G \to \mathfrak{g}^*$ is hence given by

$$\langle \mathbf{J}_L(\alpha_g), \xi \rangle = \left\langle \alpha_g, \xi_G^L(g) \right\rangle = \langle \alpha_g, T_e R_g \xi \rangle = \langle T_e^* R_g \alpha_g, \xi \rangle$$

Hence $\mathbf{J}_L(\alpha_g) = T_e^* R_g \alpha_g$.

For the cotangent lift of right translation, $\xi_G^R(g) = T_e L_g \xi$ and $\mathbf{J}_R(\alpha_g) = T_e^* L_g \alpha_g$.

Example: symplectic linear actions. Let (V, ω) be a symplectic linear space and let G be a subgroup of the linear symplectic group, acting naturally on V.

$$\langle \mathbf{J}(v), \xi \rangle = \frac{1}{2}\omega(\xi_V(v), v).$$

This $\mathbf{J}$ is not that of a cotangent lifted action.

Example: Cayley-Klein parameters and the Hopf fibration. Consider the natural action of SU(2) on $\mathbb{C}^2$. The symplectic form on $\mathbb{C}^2$ is minus the imaginary part of the Hermitian inner product.

Since this action is by isometries of the Hermitian metric, it is automatically symplectic and therefore has a momentum map $\mathbf{J} : \mathbb{C}^2 \to \mathfrak{su}(2)^*$ given, as above, by

$$\langle \mathbf{J}(z,w), \xi \rangle = \frac{1}{2}\omega\left(\xi(z,w)^{\mathsf{T}}, (z,w)^{\mathsf{T}}\right), \qquad z, w \in \mathbb{C},\ \xi \in \mathfrak{su}(2).$$

The Lie algebra $\mathfrak{su}(2)$ of SU(2) consists of 2×2 skew Hermitian matrices of trace zero. This Lie algebra is isomorphic to $\mathfrak{so}(3)$ and therefore to $(\mathbb{R}^3, \times)$ by the isomorphism given by

$$\mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3 \longmapsto \\ \tilde{\mathbf{x}} := \frac{1}{2}\begin{bmatrix} -ix^3 & -ix^1 - x^2 \\ -ix^1 + x^2 & ix^3 \end{bmatrix} \in \mathfrak{su}(2).$$

Thus we have

$$[\tilde{\mathbf{x}}, \tilde{\mathbf{y}}] = (\mathbf{x} \times \mathbf{y})^{\sim}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^3.$$

Other useful formulas are

$$\det(2\tilde{\mathbf{x}}) = \|\mathbf{x}\|^2 \quad \text{and} \quad \operatorname{trace}(\tilde{\mathbf{x}}\tilde{\mathbf{y}}) = -\frac{1}{2}\mathbf{x} \cdot \mathbf{y}.$$

Identify $\mathfrak{su}(2)^*$ with $\mathbb{R}^3$ by the map $\mu \in \mathfrak{su}(2)^* \mapsto \breve{\mu} \in \mathbb{R}^3$ defined by

$$\breve{\mu} \cdot \mathbf{x} := -2\langle \mu, \tilde{\mathbf{x}} \rangle$$

for any $\mathbf{x} \in \mathbb{R}^3$.

The symplectic form on $\mathbb{C}^2$ is given by minus the imaginary part of the Hermitian inner product.

With these notations, the momentum map $\check{\mathbf{J}} : \mathbb{C}^2 \to \mathbb{R}^3$ can be explicitly computed in coordinates: for any $\mathbf{x} \in \mathbb{R}^3$ we have

$$
\begin{aligned}
\check{\mathbf{J}}(z,w)\cdot\mathbf{x} &= -2\langle \mathbf{J}(z,w), \tilde{\mathbf{x}}\rangle \\
&= \frac{1}{2}\,\mathrm{Im}\left(\begin{bmatrix} -ix^3 & -ix^1 - x^2 \\ -ix^1 + x^2 & ix^3 \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix}\cdot\begin{bmatrix} z \\ w \end{bmatrix}\right) \\
&= -\frac{1}{2}(2\,\mathrm{Re}(w\bar{z}), 2\,\mathrm{Im}(w\bar{z}), |z|^2 - |w|^2)\cdot\mathbf{x}.
\end{aligned}
$$

Therefore

$$
\check{\mathbf{J}}(z,w) = -\frac{1}{2}(2w\bar{z}, |z|^2 - |w|^2) \in \mathbb{R}^3.
$$

$\check{\mathbf{J}}$ is a Poisson map from $\mathbb{C}^2$, endowed with the canonical symplectic structure, to $\mathbb{R}^3$, endowed with the $+$ Lie Poisson structure. Therefore, $-\check{\mathbf{J}} : \mathbb{C}^2 \to \mathbb{R}^3$ is a canonical map, if $\mathbb{R}^3$ has the $-$ Lie-Poisson bracket relative to which the free rigid body equations are Hamiltonian. Pulling back the Hamiltonian

$$H(\mathbf{\Pi}) = \frac{1}{2}\mathbf{\Pi} \cdot \mathbb{I}^{-1}\mathbf{\Pi}, \qquad \mathbb{I}^{-1}\mathbf{\Pi} := \left(\frac{\Pi_1}{I_1}, \frac{\Pi_2}{I_2}, \frac{\Pi_3}{I_3}\right)$$

to $\mathbb{C}^2$ gives a Hamiltonian function (called collective) on $\mathbb{C}^2$. $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$ is the moment of inertia tensor written in a principal axis body frame of the free rigid body.

The classical Hamilton equations for this function are therefore projected by $-\check{\mathbf{J}}$ to the rigid body equations

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbb{I}^{-1}\mathbf{\Pi}.$$

In this context, the variables (z, w) are called the **Cayley-Klein parameters**. They represent a first attempt to understand the rigid body equations as a Hamiltonian system, before the introduction of Poisson manifolds. In quantum mechanics, the same variables are called the **Kustaanheimo-Stiefel coordinates**. A similar construction was carried out in fluid dynamics making the Euler equations a Hamiltonian system relative to the so-called **Clebsch variables**.

Now notice that if

$$(z, w) \in S^3 := \left\{ (z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \right\},$$

then $\| -\check{\mathbf{J}}(z, w) \| = 1/2$, so that $-\check{\mathbf{J}}|_{S^3} : S^3 \to S^2_{1/2}$, where $S^2_{1/2}$ is the sphere in $\mathbb{R}^3$ of radius 1/2.

It is also easy to see that $-\check{\mathbf{J}}|_{S^3}$ is surjective and that its fibers are circles. Indeed, given $(x^1, x^2, x^3) = (x^1 + ix^2, x^3) = (re^{i\psi}, x^3) \in S^2_{1/2}$, the inverse image of this point is

$$-\check{\mathbf{J}}^{-1}(re^{i\psi}, x^3) = \left\{ \left(e^{i\theta} \sqrt{\frac{1}{2} + x^3}, e^{i\varphi} \sqrt{\frac{1}{2} - x^3} \right) \in S^3 \,\middle|\, e^{i(\theta - \varphi + \psi)} = 1 \right\}.$$

One recognizes now that $-\check{\mathbf{J}}|_{S^3} : S^3 \to S^2_{1/2}$ is the **Hopf fibration**. In other words:

the momentum map of the SU(2)*-action on* $\mathbb{C}^2$*, the Cayley-Klein parameters, the Kustaanheimo-Stiefel coordinates, and the family of Hopf fibrations on concentric three-spheres in* $\mathbb{C}^2$ *are* **the same map**.

Constructive proof of the Lie-Poisson Reduction Theorem

• If $\xi \in \mathfrak{g}$, denote by $\xi_L \in \mathfrak{X}(G)$ the left invariant vector field whose value at e is ξ, i.e., $\xi_L(g) = T_e L_g(\xi)$, $\forall g \in G$.

$$[\xi_L, \eta_L] = [\xi, \eta]_L$$

by definition of the Lie bracket on $\mathfrak{g}$.

• Left trivialize T^*G:

$$\lambda : T^*G \ni \alpha_g \mapsto \left(g, T_e^* L_g \alpha_g\right) = (g, \mathbf{J}_R(\alpha_g)) \in G \times \mathfrak{g}^*$$

λ is an equivariant diffeomorphism relative to the lift of left translation on T^*G and the left G-action on $G \times \mathfrak{g}^*$ given by $g \cdot (h, \mu) := (gh, \mu)$. Therefore, $(T^*G)/G \cong (G \times \mathfrak{g}^*)/G = \mathfrak{g}^*$ and hence $\mathbf{J}_R : T^*G \to \mathfrak{g}^*$ is the composition of this diffeomorphism with the canonical projection $T^*G \to (T^*G)/G$. Consequently, $\mathfrak{g}^*$ inherits a Poisson structure, which we call, for the time being $\{\cdot, \cdot\}_-$, uniquely characterized by

$$\{F_1, F_2\}_- \circ \mathbf{J}_R = \{F_1 \circ \mathbf{J}_R, F_2 \circ \mathbf{J}_R\}, \quad \forall F_1, F_2 \in C^\infty(\mathfrak{g}^*).$$

GOAL: Compute this bracket.

To do this, it is enough to work with *linear* functions F_1, F_2 because the Poisson bracket depends only on the values of the differentials of the functions at each point. If F_i is linear, then $F_i(\mu) = \left\langle \mu, \frac{\delta F_i}{\delta \mu} \right\rangle$, for some constant element $\frac{\delta F_i}{\delta \mu} \in \mathfrak{g}$. If $\mu := T_e^* L_g \alpha_g \in \mathfrak{g}^*$, we get

$$
\begin{aligned}
(F_i)_L(\alpha_g) = F_i\left(T_e^* L_g \alpha_g\right) = \left\langle T_e^* L_g \alpha_g, \frac{\delta F_i}{\delta \mu} \right\rangle = \left\langle \alpha_g, T_e L_g \frac{\delta F_i}{\delta \mu} \right\rangle \\
= \left\langle \alpha_g, \left(\frac{\delta F_i}{\delta \mu} \right)_L (g) \right\rangle = \mathcal{P}\left(\left(\frac{\delta F_i}{\delta \mu} \right)_L \right) (\alpha_g)
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\{(F_1)_L, (F_2)_L\}(\mu) &= \left\{ \mathcal{P}\left(\left(\frac{\delta F_1}{\delta \mu} \right)_L \right), \mathcal{P}\left(\left(\frac{\delta F_2}{\delta \mu} \right)_L \right) \right\}(\mu) \\
&= -\mathcal{P}\left(\left[\left(\frac{\delta F_1}{\delta \mu} \right)_L, \left(\frac{\delta F_2}{\delta \mu} \right)_L \right] \right)(\mu) \\
&= -\mathcal{P}\left(\left[\frac{\delta F_1}{\delta \mu}, \frac{\delta F_2}{\delta \mu} \right]_L \right)(\mu) \\
&= -\left\langle \mu, \left[\frac{\delta F_1}{\delta \mu}, \frac{\delta F_2}{\delta \mu} \right] \right\rangle . \qquad \square
\end{aligned}
$$

This theorem and general considerations implies the following.

Lie-Poisson reduction of dynamics

Assume that $H \in C^\infty(T^*G)$ is left(right)-invariant. Then $H^\mp := H|\mathfrak{g}^*$ satisfy $H = H^- \circ \mathbf{J}_R$ and $H = H^+ \circ \mathbf{J}_L$. The flow F_t on T^*G and the flow $F_t^\mp$ of $X_{H^\mp}$ on $\mathfrak{g}^*_\mp$ are related by

$$\mathbf{J}_R \circ F_t = F_t^- \circ \mathbf{J}_R, \qquad \mathbf{J}_L \circ F_t = F_t^+ \circ \mathbf{J}_L.$$

Remember that $\mathbf{J}_L$ is conserved.

If $\alpha(t) \in T_{g(t)}G$ is an integral curve of X_H in T^*G, let $\mu(t) := \mathbf{J}_R(\alpha(t))$, $\nu(t) := \mathbf{J}_L(\alpha(t)) = \nu =$ const. Then

$$\nu = \mathrm{Ad}^*_{g(t)^{-1}}\,\mu(t).$$

Reconstruction of dynamics

Differentiate in t the previous relation:

$$0 = \mathrm{Ad}^*_{g(t)^{-1}} \left(-\,\mathrm{ad}^*_{g(t)^{-1}\dot{g}(t)}\, \mu(t) + \frac{d\mu}{dt} \right)$$

However, $\mu(t)$ satisfies the Lie-Poisson equations

$$\frac{d\mu}{dt} = \mathrm{ad}^*_{\delta H^-/\delta\mu}\, \mu \iff \mathrm{ad}^*_{-g(t)^{-1}\dot{g}(t)+\delta H^-/\delta\mu} = 0$$

A sufficient condition for this to hold is $g(t)^{-1}\dot{g}(t) = \delta H^-/\delta\mu$. So, the integral curve of the unreduced system on T^*G is found by solving:

$$\frac{d\mu(t)}{dt} = \mathrm{ad}^*_{\frac{\delta H^-}{\delta\mu}(t)}\,\mu(t), \qquad \frac{dg(t)}{dt} = T_e L_{g(t)} \frac{\delta H^-}{\delta\mu}(t)$$

and putting

$$\alpha(t) := T^*_{g(t)} L_{g(t)^{-1}} \mu(t).$$

The expression of the push forward $\lambda_* X_H \in \mathfrak{X}(G \times \mathfrak{g})$ is

$$(\lambda_* X_H)(g,\mu) = \left(T_e L_g \frac{\delta H^-}{\delta\mu}, \mu, \mathrm{ad}^*_{\frac{\delta H^-}{\delta\mu}}\,\mu \right) \in T_g G \times T_\mu \mathfrak{g}^*.$$

Long direct proof.

More precise properties of the momentum map

- Freeness of the action is equivalent to the regularity of the momentum map: range$T_m\mathbf{J} = (\mathfrak{g}_m)^\circ$.

  Proof: We have $T_mM = \{X_f(m) \mid f \in C^\infty(U)\}$, U open neighborhood of m. For any $\xi \in \mathfrak{g}$ we have

$$\left\langle T_m\mathbf{J}\left(X_f(m)\right), \xi\right\rangle = \mathbf{d}\mathbf{J}^\xi(m)\left(X_f(m)\right) = \{\mathbf{J}^\xi, f\}(m)$$

$$= -\mathbf{d}f(m)\left(X_{\mathbf{J}^\xi}(m)\right) = -\mathbf{d}f(m)\left(\xi_M(m)\right).$$

So

$$\xi \in \mathfrak{g}_m \Longleftrightarrow \xi_M(m) = 0 \Longleftrightarrow$$
$$\mathbf{d}f(m)\left(\xi_M(m)\right) = 0, \forall f \in C^\infty(U) \Longleftrightarrow$$
$$\left\langle T_m\mathbf{J}\left(X_f(m)\right), \xi\right\rangle = 0, \forall f \in C^\infty(U) \Longleftrightarrow$$
$$\xi \in (\mathrm{range}\, T_m\mathbf{J})^\circ \qquad \square$$

- $\ker T_m\mathbf{J} = (\mathfrak{g} \cdot m)^\omega$.

Proof: $v_m \in \ker T_m\mathbf{J}$ if and only if for all $\xi \in \mathfrak{g}$

$$0 = \langle T_m\mathbf{J}(v_m), \xi\rangle = \mathbf{d}\mathbf{J}^\xi(m)(v_m) = \omega(m)\left(X_{\mathbf{J}^\xi}(m), v_m\right)$$
$$= \omega(m)\left(\xi_M(m), v_m\right)$$
$$\Longleftrightarrow v_m \in (\mathfrak{g} \cdot m)^\omega \qquad \square$$

- Existence: The obstruction is the vanishing of the map

$$\begin{array}{rccc} \rho : & \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] & \longrightarrow & H^1(M,\mathbb{R}) \\ & [\xi] & \longmapsto & [\mathbf{i}_{\xi_M}\omega] \end{array}$$

- Equivariance: When is $(\mathfrak{g},[\cdot,\cdot]) \to (C^\infty(M),\{\cdot,\cdot\})$ defined by $\xi \mapsto \mathbf{J}^\xi$, $\xi \in \mathfrak{g}$, a Lie algebra homomorphism, that is,

$$\mathbf{J}^{[\xi,\eta]} = \{\mathbf{J}^\xi, \mathbf{J}^\eta\}, \quad \xi,\eta \in \mathfrak{g}.$$

Answer: if and only if

$$T_z\mathbf{J}\left(\xi_M(z)\right) = -\operatorname{ad}^*_\xi \mathbf{J}(z),$$

A momentum map that satisfies this relation in called **infinitesimally equivariant**.

Among all possible choices of momentum maps for a given action, there is at most one infinitesimally equivariant one.

Sufficient conditions: Assume $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$. By the **Whitehead lemmas**, this is the case if $\mathfrak{g}$ is semisimple.

- $\mathbf{J}$ is G-**equivariant** when

$$\mathrm{Ad}^*_{g^{-1}} \circ \mathbf{J} = \mathbf{J} \circ \Phi_g$$

- If G is compact $\mathbf{J}$ can be chosen G-equivariant

- If G is connected then infinitesimal equivariance is equivalent to equivariance.

Define the **non-equivariance one-cocycle**, or the **the Souriau cocycle**, associated to $\mathbf{J}$ is the map

$$\begin{array}{rcc} \sigma : G & \longrightarrow & \mathfrak{g}^* \\ g & \longmapsto & \mathbf{J}(\Phi_g(z)) - \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(z)). \end{array}$$

Suppose that M is connected. Then:

(i) The definition of σ does not depend on the choice of $z \in M$. M connected is a crucial hypothesis.

(ii) The mapping σ is a $\mathfrak{g}^*$-valued one-cocycle on G with respect to the coadjoint representation of G on $\mathfrak{g}^*$.

Define the **affine action** of G on $\mathfrak{g}^*$ with cocycle σ by

$$\begin{aligned} \Xi : \; G \times \mathfrak{g}^* &\longrightarrow \mathfrak{g}^* \\ (g, \, \mu) &\longmapsto \mathrm{Ad}^*_{g^{-1}} \mu + \sigma(g). \end{aligned}$$

Ξ determines a left action of G on $\mathfrak{g}^*$. The momentum map $\mathbf{J} : M \to \mathfrak{g}^*$ is equivariant with respect to the symplectic action Φ on M and the affine action Ξ on $\mathfrak{g}^*$.

The affine orbits $\mathcal{O}_\mu$ are also symplectic with G-invariant symplectic structure given by

$$\omega^{\pm}_{\mathcal{O}_\mu}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \, \eta_{\mathfrak{g}^*}(\nu)) = \pm \langle \nu, \, [\xi, \, \eta] \rangle \mp \Sigma(\xi, \, \eta),$$

where the **infinitesimal non-equivariance two-cocycle** $\Sigma \in Z^2(\mathfrak{g}, \mathbb{R})$ is given by

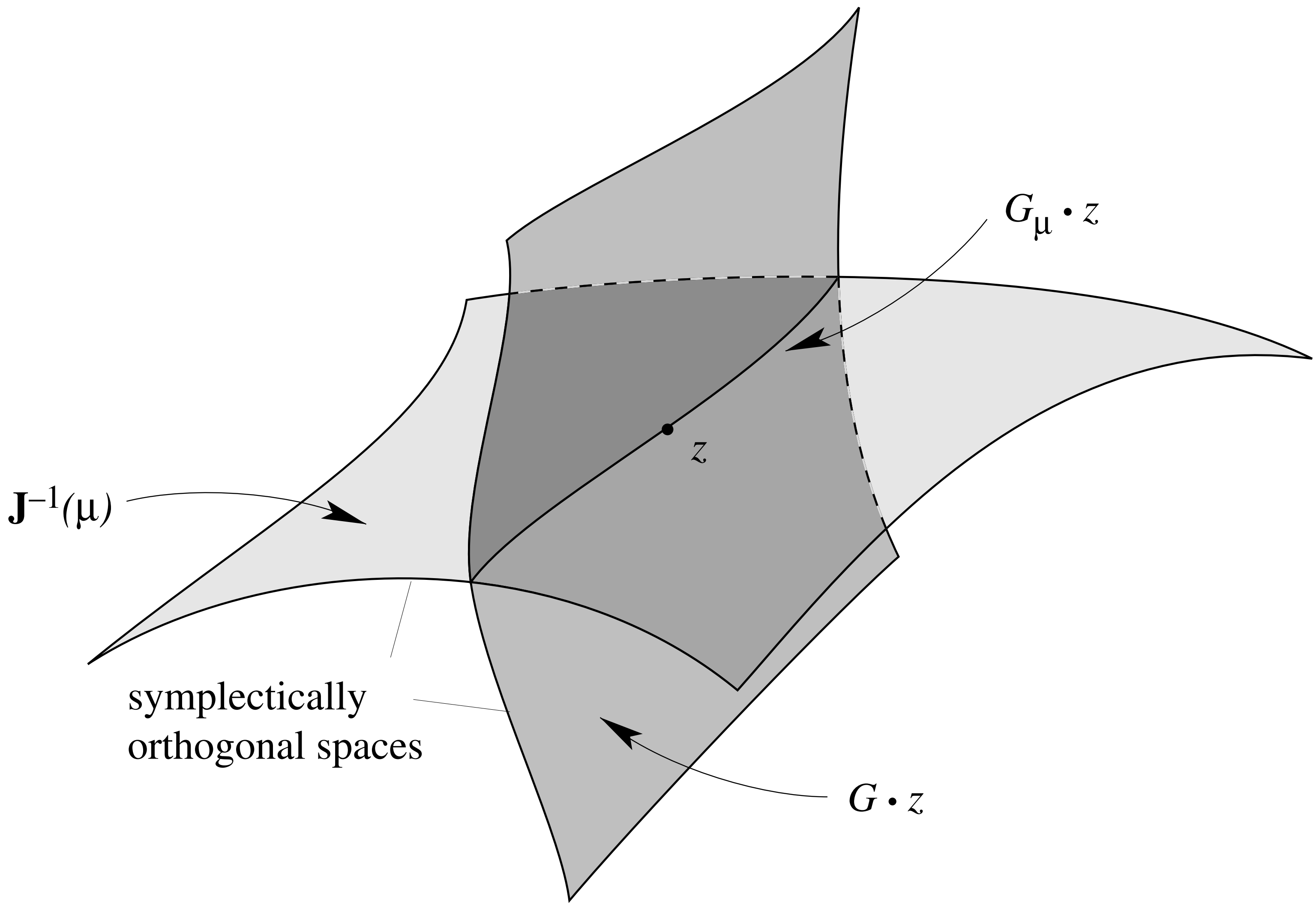
$$\begin{array}{rccc} \Sigma : & \mathfrak{g} \times \mathfrak{g} & \longrightarrow & \mathbb{R} \\ & (\xi, \eta) & \longmapsto & \Sigma(\xi, \eta) = \mathbf{d}\widehat{\sigma}_\eta(e) \cdot \xi, \end{array}$$

with $\widehat{\sigma}_\eta : G \to \mathbb{R}$ defined by $\widehat{\sigma}_\eta(g) = \langle \sigma(g), \eta \rangle$.

Reduction Lemma:

$$\mathfrak{g}_{\mathbf{J}(m)} \cdot m = \mathfrak{g} \cdot m \cap \ker T_m\mathbf{J} = \mathfrak{g} \cdot m \cap (\mathfrak{g} \cdot m)^\omega.$$

Proof: $\xi_M(m) \in \mathfrak{g} \cdot m \cap \ker T_m\mathbf{J} \iff 0 = T_m\mathbf{J}\,(\xi_M(m)) = -\operatorname{ad}^*_\xi \mathbf{J}(m) + \Sigma(\xi, \cdot) \iff \xi \in \mathfrak{g}_{\mathbf{J}(m)}$ $\square$



The geometry of the reduction lemma.

Momentum maps and isotropy type manifolds.

- $m \in M$. Then M_{G_m} is a symplectic submanifold of M.

Proof: By the Tube Theorem for proper actions, M_{G_m} is an embedded submanifold and $T_z M_{G_m} = T_z M^{G_m} = (T_z M)^{G_m}, \forall z \in M_{G_m}$. To show that $i^*\omega$ is a symplectic form, where $i : M_{G_m} \hookrightarrow M$, it suffices to show that $(i^*\omega)(z)$ is nondegenerate on $T_z M_{G_m}$, for all $z \in M_{G_m}$.

H compact Lie group and (V, ω) symplectic representation space. Then V^H is a symplectic subspace of V.

Let $\langle\!\langle\, , \rangle\!\rangle$ be a H-invariant inner product on V, possible by compactness of H (average some inner product). Define $\mathbb{T} : V \to V$ by $\langle\!\langle u, v \rangle\!\rangle = \omega(u, \mathbb{T}v)$ and note that it is a H-equivariant isomorphism. Therefore, $\mathbb{T}(V^H) \subset V^H$. Assume that $u \in V^H$ satisfies $\omega(u, v) = 0, \forall v \in V^H$. But then $0 = \omega(u, \mathbb{T}v) = \langle\!\langle u, v \rangle\!\rangle, \forall v \in V^H$. Put here $v = u$ and then the positive definiteness of $\langle\!\langle\, , \rangle\!\rangle$ implies that $u = 0$. $\square$

• Let $M^m_{G_m}$ be the connected component of M_{G_m} containing m and

$$N(G_m)^m := \{n \in N(G_m) \mid n \cdot z \in M^m_{G_m} \text{ for all } z \in M^m_{G_m}\}.$$

$N(G_m)^m$ is a closed subgroup of $N(G_m)$ that contains the connected component of the identity. So it is also open and hence $\mathrm{Lie}(N(G_m)^m) = \mathrm{Lie}(N(G_m))$.

In addition, $(N(G_m)/G_m)^m = N(G_m)^m/G_m$ so that

$$\mathrm{Lie}\,(N(G_m)^m/G_m) = \mathrm{Lie}\,(N(G_m)/G_m)\,.$$

• $L^m := N(G_m)^m/G_m$ acts freely properly and canonically on $M^m_{G_m}$ by $\Psi(nG_m, z) := n \cdot z$.

Proof: The map Ψ is clearly well defined. It is easy to see it is a left action. It is also obvious that it is free. It is proper, because $N(G_m)^m$ is closed. Still need to show that it is canonical.

For any $l = nG_m \in L^m$ we have

$$\Psi_l^*(i^*\omega) = (i \circ \Psi_l)^*\omega = (\Phi_n \circ i)^*\omega = i^*\Phi_n^*\omega = i^*\omega. \quad \square$$

• The free proper canonical action of $L^m := N(G_m)^m/G_m$ on $M^m_{G_m}$ has a momentum map $\mathbf{J}_{L^m} : M^m_{G_m} \to (\mathrm{Lie}(L^m))^*$ given by

$$\mathbf{J}_{L^m}(z) := \Lambda(\mathbf{J}|_{M^m_{G_m}}(z) - \mathbf{J}(m)), \quad z \in M^m_{G_m}.$$

In this expression $\Lambda : (\mathfrak{g}_m^\circ)^{G_m} \to (\mathrm{Lie}(L^m))^*$ denotes the natural L^m-equivariant isomorphism given by

$$\left\langle \Lambda(\beta), \left.\frac{d}{dt}\right|_{t=0} (\exp t\xi)\, G_m \right\rangle = \langle \beta, \xi \rangle,$$

for any $\beta \in (\mathfrak{g}_m^\circ)^{G_m}$, $\xi \in \mathrm{Lie}(N(G_m)^m) = \mathrm{Lie}(N(G_m))$.

- The non-equivariance one-cocycle $\tau : M^m_{G_m} \to (\mathrm{Lie}(L^m))^*$ of the momentum map $\mathbf{J}_{L^m}$ is given by the map

$$\tau(l) = \Lambda(\sigma(n) + n \cdot \mathbf{J}(m) - \mathbf{J}(m)).$$

CONVEXITY

$\mathbf{J} : M \to \mathfrak{g}^*$ coadjoint equivariant. G, M compact. The intersection of the image of $\mathbf{J}$ with a Weyl chamber is a *compact and convex polytope*. This polytope is referred to as the **momentum polytope**.

Delzant's theorem proves that the symplectic toric manifolds are classified by their momentum polytopes. A **Delzant polytope** in $\mathbb{R}^n$ is a convex polytope that is also:

(i) Simple: there are n edges meeting at each vertex.

(ii) Rational: the edges meeting at a vertex p are of the form $p + tu_i$, $0 \leq t < \infty$, $u_i \in \mathbb{Z}^n$, $i \in \{1, \ldots, n\}$.

(iii) Smooth: the vectors $\{u_1, \ldots, u_n\}$ can be chosen to be an integral basis of $\mathbb{Z}^n$.

Delzant's Theorem can be stated by saying that

$$\begin{array}{ccc} \{\text{symplectic toric manifolds}\} & \longrightarrow & \{\text{Delzant polytopes}\} \\ (M, \omega, \mathbb{T}^n, \mathbf{J} : M \to \mathbb{R}^n) & \longmapsto & \mathbf{J}(M) \end{array}$$

is a bijection.

Marsden-Weinstein Reduction Theorem

• $\mathbf{J} : M \to \mathfrak{g}^*$ equivariant (not essential)

• $\mu \in \mathbf{J}(M) \subset \mathfrak{g}^*$ regular value of $\mathbf{J}$

• G_μ-action on $\mathbf{J}^{-1}(\mu)$ is free and proper, where $G_\mu := \{g \in G \mid \mathrm{Ad}_g^* \mu = \mu\}$

then $(M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu, \omega_\mu)$ is symplectic:

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega,$$

$i_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow M$ inclusion,
$\pi_\mu : \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu)/G_\mu$ projection.

The flow F_t of X_h, $h \in C^\infty(M)^G$, leaves the connected components of $\mathbf{J}^{-1}(\mu)$ invariant and commutes with the G-action, so it induces a flow F_t^μ on M_μ by

$$\pi_\mu \circ F_t \circ i_\mu = F_t^\mu \circ \pi_\mu.$$

F_t^μ is Hamiltonian on (M_μ, ω_μ) for the **reduced Hamiltonian** $h_\mu \in C^\infty(M_\mu)$ given by

$$h_\mu \circ \pi_\mu = h \circ i_\mu.$$

Moreover, if $h, k \in C^\infty(M)^G$, then $\{h, k\}_\mu = \{h_\mu, k_\mu\}_{M_\mu}$.

Proof: Since π_μ is a surjective submersion, if ω_μ exists, it is uniquely determined by the condition $\pi_\mu^*\omega_\mu = i_\mu^*\omega$. This relation also defines ω_μ by:

$$\omega_\mu(\pi_\mu(z))\,(T_z\pi_\mu(v), T_z\pi_\mu(w)) := \omega(z)(v, w),$$

for $z \in \mathbf{J}^{-1}(\mu)$ and $v, w \in T_z\mathbf{J}^{-1}(\mu)$.

To see that this is a good definition of ω_μ, let

$$y = \Phi_g(z), \quad v' = T_z\Phi_g(v), \;\; w' = T_z\Phi_g(w)T_z\mathbf{J}^{-1}(\mu),$$

where $g \in G_\mu$. If, in addition $T_{g\cdot z}\pi_\mu(v'') = T_{g\cdot z}\pi_\mu(v') = T_z\pi_\mu(v)$ and $T_{g\cdot z}\pi_\mu(w'') = T_{g\cdot z}\pi_\mu(w') = T_z\pi_\mu(w)$, then $v'' = v' + \xi_M(g\cdot z) \in T_z\mathbf{J}^{-1}(\mu)$ and $w'' = w' + \eta_M(g\cdot z) \in T_z\mathbf{J}^{-1}(\mu)$ for some $\xi, \eta \in \mathfrak{g}_\mu$ and hence

$$
\begin{aligned}
\omega(y)(v'', w'') &= \omega(y)(v', w') \qquad \text{(by the reduction lemma)} \\
&= \omega(\Phi_g(z))(T_z\Phi_g(v), T_z\Phi_g(w)) \\
&= (\Phi_g^*\omega)(z)(v, w) \\
&= \omega(z)(v, w) \qquad \text{(action is symplectic).}
\end{aligned}
$$

Thus ω_μ is well-defined. It is smooth since $\pi_\mu^*\omega_\mu$ is smooth. Since $\mathbf{d}\omega = 0$, we get

$$
\pi_\mu^*\mathbf{d}\omega_\mu = \mathbf{d}\pi_\mu^*\omega_\mu = \mathbf{d}i_\mu^*\omega = i_\mu^*\mathbf{d}\omega = 0.
$$

Since π_μ is a surjective submersion, we conclude that $\mathbf{d}\omega_\mu = 0$.

To prove nondegeneracy of ω_μ, suppose that

$$\omega_\mu(\pi_\mu(z))(T_z\pi_\mu(v), T_z\pi_\mu(w)) = 0$$

for all $w \in T_z(\mathbf{J}^{-1}(\mu))$. This means that

$$\omega(z)(v, w) = 0 \quad \text{for all} \quad w \in T_z(\mathbf{J}^{-1}(\mu)),$$

i.e., that $v \in (T_z(\mathbf{J}^{-1}(\mu)))^\omega = T_z(G \cdot z)$ by the Reduction Lemma. Hence

$$v \in T_z(\mathbf{J}^{-1}(\mu)) \cap T_z(G \cdot z) = T_z(G_\mu \cdot z)$$

so that $T_z\pi_\mu(v) = 0$, thus proving nondegeneracy of ω_μ.

Let $Y \in \mathfrak{X}(M_\mu)$ be the vector field whose flow is F_t^μ. Therefore, from $\pi_\mu \circ F_t \circ i_\mu = F_t^\mu \circ \pi_\mu$ it follows

$$T\pi_\mu \circ X_h = Y \circ T\pi_\mu \quad \text{on} \quad \mathbf{J}^{-1}(\mu).$$

Also, $h_\mu \circ \pi_\mu = h \circ i_\mu$ implies that $\mathbf{d}h_\mu \circ T\pi_\mu = \mathbf{d}h$ on $\mathbf{J}^{-1}(\mu)$. Therefore, on $\mathbf{J}^{-1}(\mu)$ we get

$$\begin{aligned}
\pi_\mu^* (\mathbf{i}_Y \omega_\mu) &= \mathbf{i}_{X_h} \pi_\mu^* \omega_\mu = \mathbf{i}_{X_h} i_\mu^* \omega = i_\mu^* \left(\mathbf{i}_{X_h} \omega \right) = i_\mu^* \mathbf{d}h \\
&= \mathbf{d}(h \circ i_\mu) = \mathbf{d}(h_\mu \circ \pi_\mu) = \pi_\mu^* \mathbf{d}h_\mu \\
&= \pi_\mu^* \left(\mathbf{i}_{X_{h_\mu}} \omega_\mu \right),
\end{aligned}$$

so $\mathbf{i}_Y \omega_\mu = \mathbf{i}_{X_{h_\mu}} \omega_\mu$ since π_μ is a surjective submersion. Hence $Y = X_{h_\mu}$ because ω_μ is nondegenerate.

Finally, for $m \in \mathbf{J}^{-1}(\mu)$ we have

$$
\begin{aligned}
\{h_\mu, k_\mu\}_{M_\mu}(\pi_\mu(m)) &= \omega_\mu(\pi_\mu(m)) \left(X_{h_\mu}(\pi_\mu(m)), X_{k_\mu}(\pi_\mu(m)) \right) \\
&= \omega_\mu(\pi_\mu(m)) \left(T_m\pi_\mu(X_h(m)), T_m\pi_\mu(X_k(m)) \right) \\
&= (\pi_\mu^* \omega_\mu)(m) \left(X_h(m), X_k(m) \right) \\
&= (i_\mu^* \omega)(m) \left(X_h(m), X_k(m) \right) \\
&= \omega(m) \left(X_h(m), X_k(m) \right) \\
&= \{h, k\}(m) \\
&= \{h, k\}_\mu(\pi_\mu(m)),
\end{aligned}
$$

which shows that $\{h_\mu, k_\mu\}_{M_\mu} = \{h, k\}_\mu$. $\square$

Problems with the reduction procedure

- Momentum map inexistent
- How does one recover the conservation of isotropy?
- M_μ is not a smooth manifold
- G is discrete so momentum map is zero
- M is not a symplectic but a Poisson manifold

ORBIT REDUCTION

Same set up as in the symplectic reduction theorem: M connected, G acting symplectically, freely, and properly on M with an equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$.

The connected components of the point reduced spaces M_μ can be regarded as the symplectic leaves of the Poisson manifold $\left(M/G, \{\cdot,\cdot\}_{M/G}\right)$ in the following way. Form a map $[i_\mu] : M_\mu \to M/G$ defined by selecting an equivalence class $[z]_{G_\mu} \in M_\mu$ for $z \in \mathbf{J}^{-1}(\mu)$ and sending it to the class $[z]_G \in M/G$. This map is checked to be well-defined and smooth.

We then have the commutative diagram

$$\begin{array}{ccc} \mathbf{J}^{-1}(\mu) & \xrightarrow{\ i_\mu\ } & M \\ \Big\downarrow{\scriptstyle \pi_\mu} & & \Big\downarrow{\scriptstyle \pi} \\ M_\mu & \xrightarrow{\ [i_\mu]\ } & M/G \end{array}$$

One then checks that $[i_\mu]$ is a Poisson injective immersion. Moreover, the $[i_\mu]$-images in M/G of the connected components of the symplectic manifolds (M_μ, Ω_μ) are its symplectic leaves. As sets,

$$[i_\mu]\,(M_\mu) = \mathbf{J}^{-1}\,(\mathcal{O}_\mu)\,/G,$$

where $\mathcal{O}_\mu \subset \mathfrak{g}^*$ is the coadjoint orbit through $\mu \in \mathfrak{g}^*$.

$$M_{\mathcal{O}_\mu} := \mathbf{J}^{-1}\,(\mathcal{O}_\mu)\,/G$$

is called the **orbit reduced space** associated to the orbit $\mathcal{O}_\mu$. The smooth manifold structure (and hence the topology) on $M_{\mathcal{O}_\mu}$ is the one that makes

$$[i_\mu] : M_\mu \to M_{\mathcal{O}_\mu}$$

into a diffeomorphism.

An injectively immersed submanifold of S of Q is called an **initial submanifold** of Q if for any smooth manifold P, a map $g : P \to S$ is smooth if and only if $\iota \circ g : P \to Q$ is smooth, where $\iota : S \hookrightarrow Q$ is the inclusion.

Most prop. of submanifolds hold for initial submanifolds.

Symplectic Orbit Reduction Theorem

• The momentum map $\mathbf{J}$ is transverse to the coadjoint orbit $\mathcal{O}_\mu$ and hence $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ is an initial submanifold of M. Moreover, the projection $\pi_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \to M_{\mathcal{O}_\mu}$ is a surjective submersion.

• $M_{\mathcal{O}_\mu}$ is a symplectic manifold with the symplectic form $\Omega_{\mathcal{O}\mu}$ uniquely characterized by the relation

$$\pi^*_{\mathcal{O}_\mu}\Omega_{\mathcal{O}\mu} = \mathbf{J}^*_{\mathcal{O}_\mu}\omega^-_{\mathcal{O}_\mu} + i^*_{\mathcal{O}_\mu}\Omega,$$

where $\mathbf{J}_{\mathcal{O}_\mu}$ is the restriction of $\mathbf{J}$ to $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ and $i_{\mathcal{O}_\mu}$: $\mathbf{J}^{-1}(\mathcal{O}_\mu) \hookrightarrow M$ is the inclusion.

• The map $[i_\mu] : M_\mu \to M_{\mathcal{O}_\mu}$ is a symplectic diffeomorphism.

• Let h be a G-invariant function on M and define $\tilde{h}$: $M/G \to \mathbb{R}$ by $h = \tilde{h} \circ \pi$. Then the Hamiltonian vector field X_h is also G-invariant and hence induces a vector field on M/G, which coincides with the Hamiltonian vector field $X_{\widetilde{h}}$. Moreover, the flow of $X_{\widetilde{h}}$ leaves the symplectic leaves $M_{\mathcal{O}_\mu}$ of M/G invariant. This flow restricted to the symplectic leaves is again Hamiltonian relative to the symplectic form $\Omega_{\mathcal{O}\mu}$ and the Hamiltonian function $h_{\mathcal{O}_\mu}$ given by

$$h_{\mathcal{O}_\mu} \circ \pi_{\mathcal{O}_\mu} = h \circ i_{\mathcal{O}_\mu} \Longleftrightarrow h_{\mathcal{O}_\mu} = \widetilde{h}|_{\mathcal{O}_\mu}.$$

- If $h, k \in C^\infty(M)^G$, then

$$\{h, k\}_{\mathcal{O}_\mu} = \{h_{\mathcal{O}_\mu}, k_{\mathcal{O}_\mu}\}_{M_{\mathcal{O}_\mu}}.$$

This is a theorem in the Poisson category.

COTANGENT BUNDLE REDUCTION

NOTATIONS AND DEFINITIONS

Given is a smooth free proper action $\Phi : G \times Q \to Q$ and then lift the action to T^*Q; it preserves the one-form and has an equivarant momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ given by

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \alpha_q \left(\xi_Q(q) \right), \quad \text{for all} \quad \xi \in \mathfrak{g}.$$

A **connection one-form** $\mathcal{A} \in \Omega^1(Q; \mathfrak{g})$ on the principal bundle $\pi : Q \to Q/G$ satisfies

- $\mathcal{A}(q) \left(\xi_Q(q) \right) = \xi$ for all $\xi \in \mathfrak{g}$
- $\Phi_g^* \mathcal{A} = \mathrm{Ad}_g \circ \mathcal{A} \Longleftrightarrow \mathcal{A}(g \cdot q)(g \cdot v_q) = \mathrm{Ad}_g \left(\mathcal{A}(q)(v_q) \right)$

The **horizontal bundle** $H := \ker \mathcal{A}$; $TQ = H \oplus V$, where $V_q := \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}$ is the **vertical space** at $q \in Q$. We have $T_q\Phi_g(H_q) = H_{g\cdot q}$ for all $g \in G$ and $q \in Q$. The horizontal bundle characterizes the connection.

The **curvature** $\mathcal{B} = \mathrm{Curv}_{\mathcal{A}} \in \Omega^2(Q; \mathfrak{g})$ of $\mathcal{A}$ is defined by $\mathcal{B}(q)(u_q, v_q) := \mathbf{d}\mathcal{A}(q)\,(\mathrm{Hor}_q\, u_q, \mathrm{Hor}_q\, v_q)$, where $\mathrm{Hor}_q\, u_q$ is the horizontal component of u_q. The **Cartan structure equations** state

$$\mathcal{B}(X,Y) = \mathbf{d}\mathcal{A}(X,Y) - [\mathcal{A}(X), \mathcal{A}(Y)] \text{ for all } X, Y \in \mathfrak{X}(Q).$$

COTANGENT BUNDLE REDUCTION: EMBEDDING VERSION

What is $(T^*Q)_\mu$ concretely?

Form the left principal G_μ-bundle $\pi_{Q,G_\mu} : Q \to Q_\mu := Q/G_\mu$.The momentum map $\mathbf{J}^\mu : T^*Q \to \mathfrak{g}_\mu^*$ is

$$\mathbf{J}^\mu(\alpha_q) = \mathbf{J}(\alpha_q)|_{\mathfrak{g}_\mu}$$

Let $\mu' := \mu|_{\mathfrak{g}_\mu} \in \mathfrak{g}_\mu^*$. Notice that there is a natural inclusion of submanifolds

$$\mathbf{J}^{-1}(\mu) \subset (\mathbf{J}^\mu)^{-1}(\mu').$$

Since the actions are free and proper, μ and μ' are regular values, so these sets are indeed smooth manifolds. Note that, by construction, μ' is G_μ-invariant.

There will be two key assumptions relevant to the embedding version of cotangent bundle reduction. Namely,

CBR1. *In the above setting, assume there is a G_μ-invariant one-form α_μ on Q with values in $(\mathbf{J}^\mu)^{-1}(\mu')$.*

and the stronger condition

CBR2. *Assume that α_μ in* **CBR1** *takes values in $\mathbf{J}^{-1}(\mu)$.*

Then there is a unique two-form β_μ on Q_μ such that

$$\pi^*_{Q,G_\mu}\beta_\mu = \mathbf{d}\alpha_\mu.$$

Since π_{Q,G_μ} is a submersion, β_μ is closed (it need not be exact). Let

$$B_\mu = \pi^*_{Q_\mu}\beta_\mu \in \Omega^2(T^*Q_\mu),$$

where $\pi_{Q_\mu} : T^*Q_\mu \to Q_\mu$ is the cotangent bundle projection. Also, to avoid confusion with the canonical symplectic form Ω_{can} on T^*Q, we shall denote the canonical symplectic form on T^*Q_μ, the cotangent bundle of μ-shape space, by ω_{can}.

- If condition **CBR1** holds, then there is a symplectic embedding

$$\varphi_\mu : ((T^*Q)_\mu, \Omega_\mu) \to (T^*Q_\mu, \omega_{\text{can}} - B_\mu),$$

onto a submanifold of T^*Q_μ covering the base Q/G_μ.

- This map φ_μ gives a symplectic diffeomorphism of $((T^*Q)_\mu, \Omega_\mu)$ onto $(T^*Q_\mu, \omega_{\text{can}} - B_\mu)$ if and only if $\mathfrak{g} = \mathfrak{g}_\mu$.

- If **CBR2** holds, then the image of φ_μ equals the vector subbundle $[T\pi_{Q,G_\mu}(V)]^\circ$ of T^*Q_μ, where $V \subset TQ$ is the vector subbundle consisting of vectors tangent to the G-orbits, that is, its fiber at $q \in Q$ equals $V_q = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}$, and $^\circ$ denotes the annihilator relative to the natural duality pairing between TQ_μ and T^*Q_μ.

- Assume that $\mathcal{A} \in \Omega^1(Q; \mathfrak{g})$ is a connection on the principal bundle $\pi_{Q,G} : Q \to Q/G$. Then $\alpha_\mu(q) := \langle \mu, \mathcal{A}(q) \rangle = \mathcal{A}(q)^*\mu \in \Omega^1(Q)$ satisfies **CBR2**. This implies that B_μ is the pull back to T^*Q_μ of $\mathbf{d}\alpha_\mu \in \Omega^2(Q)$, which equals the μ-component of the two form $\mathcal{B} + [\mathcal{A}, \mathcal{A}] \in \Omega^2(Q; \mathfrak{g})$, where $\mathcal{B}$ is the curvature of $\mathcal{A}$.

COTANGENT BUNDLE REDUCTION: BUNDLE VERSION

Again we will utilize a choice of connection $\mathcal{A}$ on the shape space bundle $\pi_{Q,G} : Q \to Q/G$. A key step in the argument is to utilize orbit reduction and the identification $(T^*Q)_\mu \cong (T^*Q)_{\mathcal{O}}$. Q/G is called the **shape space**.

The reduced space $(T^*Q)_\mu$ is a locally trivial fiber bundle over $T^*(Q/G)$ with typical fiber $\mathcal{O}$:

$$(T^*Q)_\mu \xrightarrow{\mathcal{O}} T^*(Q/G)$$

ASSOCIATED BUNDLES

G also acts on a manifold V on the left. Then $g\cdot(q,v) := (g\cdot q, g\cdot v)$ is a free proper action so form $P\times_G V := (P\times \times V)/G$. This is a locally trivial fiber bundle over Q/G all of whose fibers are diffeomorphic to V.

If V is a representation space of G, then $Q\times_G V \to Q/G$ is a vector bundle. In particular, if V is $\mathfrak{g}$ or $\mathfrak{g}^*$ and the G-action is the adjoint or coadjoint action, then $\tilde{\mathfrak{g}} := Q\times_G \mathfrak{g}$ is the **adjoint bundle** and its dual $\tilde{\mathfrak{g}}^* := Q\times_G \mathfrak{g}^*$ is the **coadjoint bundle**.

Unlike the connection form $\mathcal{A}$, the curvature drops to an adjoint bundle valued two-form $\bar{\mathcal{B}}$ on the base Q/G, namely,

$$\bar{\mathcal{B}}(\pi(q))\,(T_q\pi(u_q), T_q\pi(v_q)) := [q, \mathcal{B}(q)(u_q, v_q)] \in \tilde{\mathfrak{g}}$$

PULL BACK COMMUTES WITH ASSOCIATING

• $\pi : P \to M$ left principal G-bundle. $\tau : N \to M$ surjective submersion. Define the **pull back bundle** over N by

$$\tilde{P} := \{(n, p) \in N \times P \mid \pi(p) = \tau(n)\}.$$

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\tilde{\tau}_{N,P}} & P \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ N & \xrightarrow{\tau} & M \end{array}$$

$\tilde{\pi} : \tilde{P} \to N$ and $\tilde{\tau}_{N,P} : \tilde{P} \to P$ are the projections on the first and second factors. $\tilde{P}$ is a smooth manifold of dimension $\dim P + \dim N - \dim M$ and the free G-action on P induces a free G-action on $\tilde{P}$ given by

$$g \cdot (n, p) = (n, gp)$$

with respect to which, $\tilde{\pi}$ is the projection on the space of orbits.

$\tilde{P}$ is a left principal G-bundle over N and the map $\tilde{\tau}_{N,P}$ is a submersion with fiber over the point $p \in P$ equal to

$$\begin{aligned}\tilde{\tau}_{N,P}^{-1}(p) &= \{(n,p) \in N \times P \mid \pi(p) = \tau(n)\} \\ &= \tau^{-1}(\pi(p)) \times \{p\} \subset \tilde{P}\end{aligned}$$

and hence diffeomorphic to $\tau^{-1}(\pi(p))$.

Now suppose that there is a left action of G on a manifold V. There are two associated bundles that one can construct: $P \times_G V$ and $\tilde{P} \times_G V$. They are fiber bundles over M and N respectively, both with fibers diffeomorphic to V.

The associated bundle $\tilde{P} \times_G V \to N$ is obtained from the principal bundle $\pi : P \to M$, the surjective submersion $\tau : N \to M$, and the G-manifold V by pull back and association.

These operations can be reversed. First one forms the associated bundle $\pi_E : [p, v] \in E := P \times_G V \mapsto \pi(p) \in M$ and then one pulls it back by the surjective submersion $\tau : N \to M$. One obtains the pull back bundle $\tilde{\pi}_E : \tilde{E} \to N$, whose fibers are all diffeomorphic to V, defined by the following commutative diagram

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{\tau}_{N,E}} & E = P \times_G V \\ \Big\downarrow{\scriptstyle \tilde{\pi}_E} & & \Big\downarrow{\scriptstyle \pi_E} \\ N & \xrightarrow{\tau} & M \end{array}$$

$\tilde{E} := \{(n, [p, v]) \mid \tau(n) = \pi_E([p, v]) = \pi(p)\}$

$\tilde{\pi}_E(n, [p, v]) := n, \ \tilde{\tau}_{N,E}(n, [p, v]) := [p, v].$

The fibers of $\tilde{\tau}_{N,E}$ are equal to

$$\begin{aligned} \tilde{\tau}_{N,E}^{-1}([p, v]) &= \{(n, [p, v]) \mid \tau(n) = \pi_E([p, v]) = \pi(p)\} \\ &= \tau^{-1}(\pi(p)) \times \{[p, v]\} \simeq \tau^{-1}(\pi(p)). \end{aligned}$$

There is a canonical bundle isomorphism over M

$$[(n,p),v] \in \tilde{P} \times_G V \longrightarrow (n,[p,v]) \in \tilde{E}.$$

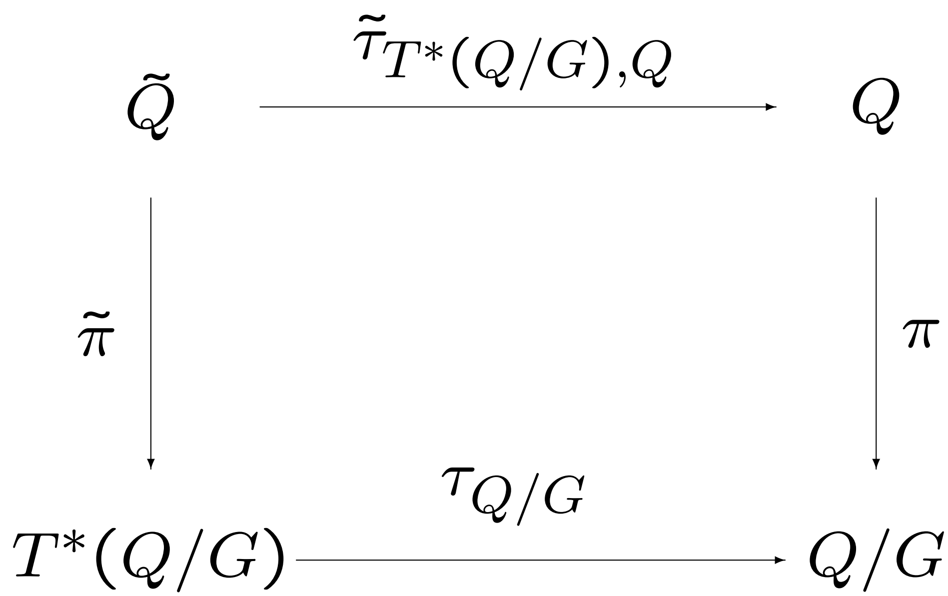
STERNBERG SPACE

$G \times Q \to Q$ free proper action, $\pi : Q \to Q/G$
$\mathcal{A} \in \Omega^1(Q;\mathfrak{g})$ connection, $V(Q)$, $H(Q)$ vertical and horizontal subbundles of TQ, $V_q(Q) = \ker T_q\pi$, $H_q(Q) = \ker \mathcal{A}(q)$, $TQ = V(Q) \oplus H(Q)$.

Pull back $\pi : Q \to Q/G$ by the cotangent bundle projection $\tau_{Q/G} : T^*(Q/G) \to Q/G$ to get the G-principal bundle

$$\tilde{Q} = \{(\alpha_{[q]}, q) \in T^*(Q/G) \times Q \mid [q] = \pi(q), q \in Q\}$$

over $T^*(Q/G)$ with fiber over $\alpha_{[q]}$ diffeomorphic to $\pi^{-1}([q])$. Recall that the G-action on $\tilde{Q}$ is given by $g \cdot (\alpha_{[q]}, q) := (\alpha_{[q]}, g \cdot q)$ for any $g \in G$ and $(\alpha_{[q]}, q) \in \tilde{Q}$.



$\tilde{Q}$ is a vector bundle over Q which is isomorphic to the annihilator $V(Q)^\circ \subset T^*Q$ of $V(Q) \subset TQ$. For each $q \in Q$,

$$V_q(Q)^\circ := \{\alpha_q \in T_q^*Q \mid \left\langle \alpha_q, \xi_Q(q) \right\rangle = 0\} \subset T_q^*Q$$

Form the coadjoint bundle of $\tilde{Q}$, the **Sternberg space**

$$S := \tilde{Q} \times_G \mathfrak{g}^*.$$

The map $\varphi_{\mathcal{A}} : \tilde{Q} \times \mathfrak{g}^* \to T^*Q$ given by

$$\varphi_{\mathcal{A}} \left(\left(\alpha_{[q]}, q \right), \mu \right) := T_q^*\pi(\alpha_{[q]}) + \mathcal{A}(q)^*\mu$$

is a G-equivariant vector bundle isomorphism over Q. It descends to a vector bundle isomorphism over Q/G

$$\Phi_{\mathcal{A}} : S \to (T^*Q)/G.$$

The **Sternberg space Poisson bracket** $\{\cdot,\cdot\}_S$ is defined as the pull back by $\Phi_{\mathcal{A}}$ of the Poisson bracket of $(T^*Q)/G$.

WEINSTEIN SPACE

Form the coadjoint bundle $\tilde{\mathfrak{g}}^* := Q \times_G \mathfrak{g}^*$. Then pull it back by the cotangent bundle projection $\tau_{Q/G} : T^*(Q/G) \to Q/G$ and get

$$W := \{(\alpha_{[q]}, [q, \mu]) \in T^*(Q/G) \times \tilde{\mathfrak{g}}^* \mid \tau_{Q/G}(\alpha_{[q]}) = \pi_{\tilde{\mathfrak{g}}^*}([q, \mu]) := [q]\}$$

$$
\begin{array}{ccc}
W & \xrightarrow{\tilde{\tau}_{T^*(Q/G),\widetilde{\mathfrak{g}}^*}} & \widetilde{\mathfrak{g}}^* \\
\Big\downarrow \tilde{\pi}_{\widetilde{\mathfrak{g}}^*} & & \Big\downarrow \pi_{\widetilde{\mathfrak{g}}^*} \\
T^*(Q/G) & \xrightarrow{\tau_{Q/G}} & Q/G
\end{array}
$$

$\tilde{\pi}_{\widetilde{\mathfrak{g}}^*}$, $\tilde{\tau}_{T^*(Q/G),\widetilde{\mathfrak{g}}^*}$ first and second projections.

W is a vector bundle over $T^*(Q/G)$ with fiber $\tilde{\pi}_{\widetilde{\mathfrak{g}}^*}^{-1}(\alpha_{[q]}) = \pi_{\widetilde{\mathfrak{g}}^*}^{-1}([q]) = \{[q,\mu] \mid \mu \in \mathfrak{g}^*\}$ over $\alpha_{[q]}$.

W is also a vector bundle over Q/G relative to the projection $(\alpha_{[q]}, [q,\mu]) \in W \mapsto [q] \in Q/G$; the fiber over $[q]$

equals $W_{[q]} = T^*_{[q]}(Q/G) \oplus \tilde{\mathfrak{g}}^*_{[q]}$. That is, we have the immediate identification

$$W = T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

as vector bundles of Q/G.

There exists a vector bundle isomorphism over Q/G

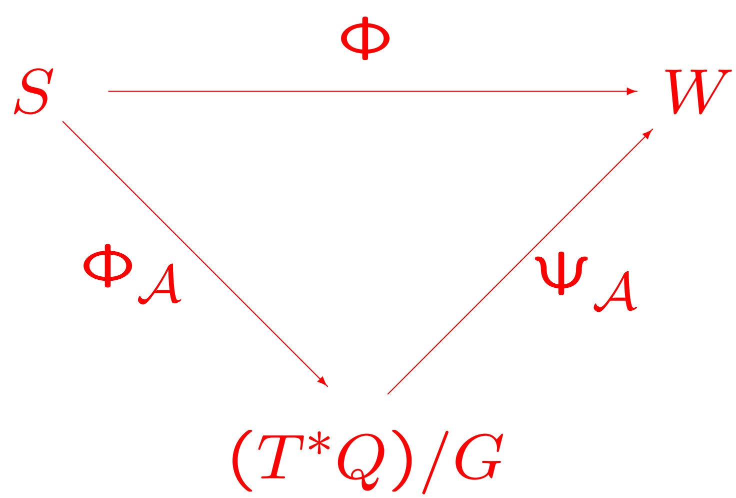
$$\Psi_{\mathcal{A}} : [\alpha_q] \in (T^*Q)/G \longmapsto (\mathrm{hor}^*_q(\alpha_q), [q, \mathbf{J}(\alpha_q)]) \in W,$$

where $\mathrm{hor}_q := (T_q\pi|_{H(Q)_q})^{-1} : T_{[q]}(Q/G) \to H_q(Q) \subset T_qQ$ is the horizontal lift operator. Thus $\mathrm{hor}^*_q : T^*_qQ \to T^*_{[q]}(Q/G)$ is a linear surjective map whose kernel is the annihilator $H(Q)^\circ_q$ of the horizontal space.

The **Weinstein space Poisson bracket** $\{\cdot,\cdot\}_W$ is the push forward by $\Psi_{\mathcal{A}}$ of the Poisson bracket of $(T^*Q)/G$.

Recall that pull back and association commute.

The following diagram of vector bundle isomorphisms over Q/G is commutative



$\Phi : (S, \{\cdot,\cdot\}_S) \to (W, \{\cdot,\cdot\}_W)$ is an isomorphism of Poisson manifolds. Also, $\Phi^* : W^*_{\alpha_{[q]}} \to S^*_{\alpha_{[q]}}$ restricted to each fiber (which is isomorphic to $\mathfrak{g}$) is an isomorphism of Lie algebras for every $\alpha_{[q]} \in T^*(Q/G)$, that is, $\Phi^* : W^* \to S^*$ is an isomorphism of Lie algebra bundles.

COVARIANT EXTERIOR DERIVATIVES ON ASSOCIATED BUNDLES

$\pi : P \to M$ left principal G-bundle, V a left representation space of G, $\mathrm{hor}_p : T_{\pi(p)}M \to T_pP$ the horizontal lift operator at $p \in P$ of the given connection $\mathcal{A} \in \Omega^1(P; \mathfrak{g})$. Then the horizontal lift operator of the induced affine connection on the associated vector bundle $\pi_E : E = P \times_G V \to M$ induced by $\mathcal{A}$ is given by

$$\mathrm{hor}_{[p,v]}(u_m) := T_{(p,v)}\pi_{P\times V}\,(\mathrm{hor}_p(u_m), 0)\,,$$

where $p \in P$, $v \in V$, $m = \pi(p) = [p]$, $u_m \in T_mM$, $\pi_{P\times V} : P \times V \to E$ is the orbit map, and $[p,v] := \pi_{P\times V}(p,v) \in E$.

The covariant derivative $\mathbf{d}_{\mathcal{A}}f$ of $f \in C^\infty(P \times_G V)$ relative to the affine connection given by this horizontal lift operator is

$$\mathbf{d}_{\mathcal{A}}f([p,v])(u_m) := \mathbf{d}f([p,v])\left(\mathrm{hor}_{[p,v]}(u_m)\right) \in T^*_mM.$$

COVARIANT EXTERIOR DERIVATIVES ON PULL BACK VECTOR BUNDLES

$\pi : E \to M$ vector bundle with an affine connection ∇, N another manifold, $\tau : N \to M$ a surjective submersion. Denote by $\tilde{E} := \{(n, \epsilon) \mid \tau(n) = \pi(\epsilon)\}$ the pull back bundle over N, which is a vector bundle $\tilde{\pi} : \tilde{E} \to N$, where $\tilde{\pi}$ is the projection on the first factor N. Denote by $\tilde{\tau}_{N,E} : \tilde{E} \to E$ the projection on the second factor E and recall that $\pi \circ \tilde{\tau}_{N,E} = \tau \circ \tilde{\pi}$. Denote for any $\epsilon \in E$ by $\mathrm{hor}_\epsilon : T_{\pi(\epsilon)}M \to T_\epsilon E$ the horizontal lift operator of the connection ∇.

Define the horizontal lift operator $\text{hor}_{(n,\epsilon)} : T_nN \to T_{(n,\epsilon)}\tilde{E}$

$$\text{hor}_{(n,\epsilon)}(v_n) := (v_n, \text{hor}_\epsilon\, T_n\tau(v_n))$$

for $(n,\epsilon) \in \tilde{E}$, $v_n \in T_nN$.

If $f \in C^\infty(\tilde{E})$, its covariant exterior derivative $\widetilde{\nabla} f(n,\epsilon) \in T_n^*N$ is defined by

$$\widetilde{\nabla} f(n,\epsilon)(v_n) := \mathbf{d}f(n,\epsilon)\big(\,\text{hor}_{(n,\epsilon)}(v_n)\big),$$

where $(n,\epsilon) \in \tilde{P}$ and $v_n \in T_nN$.

COVARIANT EXTERIOR DERIVATIVES ON S AND W

Recall that $\tilde{\pi} : \tilde{Q} \to T^*(Q/G)$ is a principal G-bundle, the pull back of $\pi : Q \to Q/G$ over the cotangent bundle projection $\tau_{Q/G} : T^*(Q/G) \to Q/G$. Recall that $\tilde{\tau}_{T^*(Q/G),Q} : \tilde{Q} \to Q$ is the projection on the seccond factor. So $\tilde{\mathcal{A}} := \tilde{\tau}^*_{T^*(Q/G),Q}\mathcal{A} \in \Omega^1(\tilde{Q}; \mathfrak{g})$ is a connection. Its horizontal lift is

$$\mathrm{hor}_{(\alpha_{[q]},q)}\left(v_{\alpha_{[q]}}\right) = \left(v_{\alpha_{[q]}}, \mathrm{hor}_q\left(T_{\alpha_{[q]}}\tau_{Q/G}(v_{\alpha_{[q]}})\right)\right).$$

$$H_{(\alpha_{[q]},q)}(\tilde{Q}) = T_{\alpha_{[q]}}(T^*(Q/G)) \times H_q(Q).$$

For the case of the associated bundle $\tilde{\pi}_{\tilde{Q}} : S \to T^*(Q/G)$, $S := \tilde{Q} \times_G \mathfrak{g}^*$, $\tilde{\pi}_{\tilde{Q}}([(\alpha_{[q]}, q), \mu]) = \alpha_{[q]}$, the formula for the associated horizontal lift at $s = [(\alpha_{[q]}, q), \mu] \in S$ becomes

$$\begin{aligned} \mathrm{hor}_s(v_{\alpha_{[q]}}) &= T_{((\alpha_{[q]},q),\mu)} \pi_{\tilde{Q}\times\mathfrak{g}^*} \left(\mathrm{hor}_{(\alpha_{[q]},q)} v_{\alpha_{[q]}}, 0 \right) \\ &= T_{((\alpha_{[q]},q),\mu)} \pi_{\tilde{Q}\times\mathfrak{g}^*} \left(\left(v_{\alpha_{[q]}}, \mathrm{hor}_q (T_{\alpha_{[q]}} \tau_{Q/G}(v_{\alpha_{[q]}})) \right), 0 \right), \end{aligned}$$

$\pi_{\tilde{Q}\times\mathfrak{g}^*} : \tilde{Q} \times \mathfrak{g}^* \to S = \tilde{Q} \times_G \mathfrak{g}^*$ is the orbit projection.

Let $f \in C^\infty(S)$, $s = [(\alpha_{[q]}, q), \mu] \in S$. The pull back connection one-form $\tilde{\mathcal{A}} \in \Omega^1(\tilde{Q}; \mathfrak{g})$ defines hence a covector

$\mathbf{d}^S_{\tilde{\mathcal{A}}} f(s) \in T^*_{\tilde{\pi}_{\tilde{Q}}(s)} T^*(Q/G)$ by

$$\mathbf{d}^S_{\tilde{\mathcal{A}}} f(s) \left(v_{\alpha_{[q]}} \right) := \mathbf{d} f(s) \left(\mathrm{hor}_s \left(v_{\alpha_{[q]}} \right) \right) =$$
$$\mathbf{d} f(s) \left(T_{((\alpha_{[q]}, q), \mu)} \pi_{\tilde{Q} \times \mathfrak{g}^*} \left(\left(v_{\alpha_{[q]}}, \mathrm{hor}_q (T_{\alpha_{[q]}} \tau_{Q/G} (v_{\alpha_{[q]}})) \right), 0 \right) \right),$$

where $\tilde{\pi}_{\tilde{Q}}(s) = \alpha_{[q]}$, and and $v_{\alpha_{[q]}} \in T_{\alpha_{[q]}} (T^*(Q/G))$.

W is the pull back of the vector bundle $\pi_{\tilde{\mathfrak{g}}^*} : \tilde{\mathfrak{g}}^* Q/G$, which has an affine connection as an associated bundle, by $\tau_{Q/G} : T^*(Q/G) \to Q/G$. So there is an induced $\widetilde{\nabla}^W$ covariant derivative on W. If $f \in C^\infty(W)$ then

$$\widetilde{\nabla}^W f(\alpha_{[q]}, [q, \mu]) = \mathbf{d} f(\alpha_{[q]}, [q, \mu]) \circ \mathrm{hor}_{(\alpha_{[q]}, [q, \mu])}$$
$$\in T^*_{\alpha_{[q]}} (T^*(Q/G)).$$

POISSON BRACKETS ON S AND W

Let $s = [(\alpha_{[q]}, q), \mu] \in S$ and $v = [q, \mu] \in \tilde{\mathfrak{g}}^*$. The Poisson bracket of $f, g \in C^\infty(S)$ is given by

$$\{f, g\}_S(s) = \Omega_{Q/G}(\alpha_{[q]}) \left(\mathbf{d}^S_{\tilde{A}} f(s)^\sharp, \mathbf{d}^S_{\tilde{A}} g(s)^\sharp \right) + \left\langle v, \tilde{\mathcal{B}}(\alpha_{[q]}) \left(\mathbf{d}^S_{\tilde{A}} f(s)^\sharp, \mathbf{d}^S_{\tilde{A}} g(s)^\sharp \right) \right\rangle - \left\langle s, \left[\frac{\delta f}{\delta s}, \frac{\delta g}{\delta s} \right] \right\rangle,$$

where $\Omega_{Q/G}$ is the canonical symplectic form on $T^*(Q/G)$, $\tilde{\mathcal{B}} \in \Omega^2(T^*(Q/G); \tilde{\mathfrak{g}})$ is thus the $\tilde{\mathfrak{g}}$-valued two-form on $T^*(Q/G)$ given by $\tilde{\mathcal{B}} = \tau^*_{Q/G} \bar{\mathcal{B}}$, with $\bar{\mathcal{B}} \in \Omega^2(Q/G, \tilde{\mathfrak{g}})$, $\sharp : T^*(T^*(Q/G)) \to T(T^*(Q/G))$ is the vector bundle isomorphism induced by $\Omega_{Q/G}$, and $\delta f / \delta s \in S^* = \tilde{Q} \times_G \mathfrak{g}$ is the usual fiber derivative of f at the point $s \in S$, that is,

$$\left\langle s', \frac{\delta f}{\delta s} \right\rangle := \frac{d}{dt}\bigg|_{t=0} f\left([(\alpha_{[q]}, q), \mu + t\nu]\right)$$

for any $s' := [(\alpha_{[q]}, q), \nu)] \in S$.

The third term has a more convenient expression. Denote by $\delta f / \delta v \in \tilde{\mathfrak{g}}$ the unique element in the fiber at $[q]$ of the adjoint bundle $\tilde{\mathfrak{g}}$ defined by the equality

$$\begin{aligned}\left\langle [q, \nu], \frac{\delta f}{\delta v} \right\rangle &= \frac{d}{dt}\bigg|_{t=0} f\left([(\alpha_{[q]}, q), \mu + t\nu]\right) \\ &= \left\langle [(\alpha_{[q]}, q), \nu)], \frac{\delta f}{\delta s} \right\rangle\end{aligned}$$

for any $\nu \in \mathfrak{g}^*$, where $s = [(\alpha_{[q]}, q), \mu] \in S = \tilde{Q} \times_G \mathfrak{g}^*$ and $v = [q, \mu] \in \tilde{\mathfrak{g}}^*$.

Thus $\delta f/\delta v$ is an element in $\tilde{\mathfrak{g}}$ over the point $[q] \in Q/G$ and can therefore be paired with $[q, \nu] \in \tilde{\mathfrak{g}}^*$. Note that we abuse here the symbol $\delta f/\delta v$ which should denote the usual fiber derivative of a function on the vector bundle $\tilde{\mathfrak{g}}^*$; however, this makes no a priori sense in this case, since $f \in C^\infty(S)$ is not a function on $\tilde{\mathfrak{g}}^*$. Nevertheless we retain this notation for it is suggestive of the result. With this definition, for $s = [(\alpha_{[q]}, q), \mu] \in S$ and $v = [q, \mu] \in \tilde{\mathfrak{g}}^*$, we have

$$\left\langle s, \left[\frac{\delta f}{\delta s}, \frac{\delta g}{\delta s}\right]\right\rangle = \left\langle v, \left[\frac{\delta f}{\delta v}, \frac{\delta g}{\delta v}\right]\right\rangle .$$

$w = (\alpha_{[q]}, [q, \mu])$, $v = [q, \mu]$, $\tilde{\mathcal{B}} = \tau^*_{Q/G}\bar{\mathcal{B}} \in \Omega^2\left(T^*(Q/G); \tilde{\mathfrak{g}}\right)$.

The Poisson bracket of $f, g \in C^\infty(W)$ is given by

$$
\begin{aligned}
\{f, g\}_W(w) = \Omega_{Q/G}(\alpha_{[q]}) &\left(\widetilde{\nabla}^W_{\mathcal{A}} f(w)^\sharp, \widetilde{\nabla}^W_{\mathcal{A}} g(w)^\sharp\right) \\
&+ \left\langle v, \tilde{\mathcal{B}}(\alpha_{[q]}) \left(\widetilde{\nabla}^W_{\mathcal{A}} f(w)^\sharp, \widetilde{\nabla}^W_{\mathcal{A}} g(w)^\sharp\right)\right\rangle \\
&- \left\langle w, \left[\frac{\delta f}{\delta w}, \frac{\delta g}{\delta w}\right]\right\rangle .
\end{aligned}
$$

$\delta f/\delta w \in W^*$ is the fiber derivative of f in W.

What are the symplectic leaves?

MINIMAL COUPLING CONSTRUCTION

Construction of presymplectic forms on associated bundles.

$\sigma : P \to B$ a left principal G-bundle, $\mathcal{A} \in \Omega^1(P; \mathfrak{g})$ a connection one-form on P, (M, ω) a Hamiltonian G-space with equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$, and denote by $\Pi_P : P \times M \to P$ and $\Pi_M : P \times M \to M$ the two projections. Then $\left\langle \Pi_M^* \mathbf{J}, \Pi_P^* \mathcal{A} \right\rangle \in \Omega^1(P \times M)$ defined by

$$\langle \Pi_M^* \mathbf{J}, \Pi_P^* \mathcal{A} \rangle (p, m)(u_p, v_m) := \langle \mathbf{J}(m), \mathcal{A}(p)(v_p) \rangle$$

for all $p \in P, m \in M, u_p \in T_pP$, and $v_m \in T_mM$, is a G-invariant one-form.

Thus, if $\xi_{P\times M} = (\xi_P, \xi_M)$ is the infinitesimal generator of the diagonal G-action on $P \times M$ defined by $\xi \in \mathfrak{g}$, we have $\boldsymbol{\pounds}_{\xi_{P\times M}} \left\langle \Pi_M^* \mathbf{J}, \Pi_P^* \mathcal{A} \right\rangle = 0$. A computation shows

$$\mathbf{i}_{\xi_{P\times M}} \left(\mathbf{d} \left\langle \Pi_M^* \mathbf{J}, \Pi_P^* \mathcal{A} \right\rangle + \Pi_M^* \omega \right) = 0.$$

Since $\mathbf{d} \left\langle \Pi_M^* \mathbf{J}, \Pi_P^* \mathcal{A} \right\rangle + \Pi_M^* \omega$ is also G-invariant, it follows that the closed two-form $\mathbf{d} \left\langle \Pi_M^* \mathbf{J}, \Pi_P^* \mathcal{A} \right\rangle + \Pi_M^* \omega$ descends to a closed two form $\omega^{\mathcal{A}} \in \Omega^2(P \times_G M)$, that is, $\omega^{\mathcal{A}}$ is characterized by the relation

$$\rho^* \omega^{\mathcal{A}} = \mathbf{d} \left\langle \Pi_M^* \mathbf{J}, \Pi_P^* \mathcal{A} \right\rangle + \Pi_M^* \omega,$$

where $\rho : P \times M \to P \times_G M$ is the projection to the orbit space.

Now assume, in addition, that the base (B, Ω) is a symplectic manifold and denote by $\sigma_M : P \times_G M \to B$ the associated fiber bundle projection given by $\sigma_M([p, m]) := \sigma(p)$. Then $\sigma_M^* \Omega$ is also a closed two-form on $P \times_G M$ and one gets the **minimal coupling** presymplectic form $\omega^{\mathcal{A}} + \sigma_M^* \Omega$. In general, this presymplectic form is degenerate.

SYMPLECTIC FORM ON $\tilde{Q} \times_G \mathcal{O}$

Apply the minimal coupling construction: $P = \tilde{Q}$, $B = T^*(Q/G)$, $\Omega = \Omega_{Q/G} = -\mathbf{d}\Theta_{Q/G}$, $\sigma = \tilde{\pi} : (\alpha_{[q]}, q) \in \tilde{Q} \mapsto \alpha_{[q]} \in T^*(Q/G)$, the connection on this principal G-bundle is $\tilde{\mathcal{A}} = \tilde{\tau}^*_{T^*(Q/G),Q}\mathcal{A} \in \Omega^1(\tilde{Q}; \mathfrak{g})$, where $\tilde{\tau}_{T^*(Q/G),Q} : \tilde{Q} \to Q$ is the projection on the second factor, $(M, \omega) = (\mathcal{O}, \omega^-_{\mathcal{O}})$, $\mathbf{J} = \mathbf{J}_{\mathcal{O}} : \mathcal{O} \to \mathfrak{g}^*$ is given by $\mathbf{J}_{\mathcal{O}}(\mu) = -\mu$ for any $\mu \in \mathfrak{g}^*$, and $\rho : \tilde{Q} \times \mathcal{O} \to \tilde{Q} \times_G \mathcal{O}$ is the quotient map for the diagonal G-action. Note that $\rho = \pi_{\tilde{Q}\times\mathfrak{g}^*}|_{\tilde{Q}\times\mathcal{O}}$ where $\pi_{\tilde{Q}\times\mathfrak{g}^*} : \tilde{Q} \times \mathfrak{g}^* \to S$ is the projection onto the G-orbit space. Then $\sigma_M = \tilde{\pi}_{\tilde{Q}} : \tilde{Q} \times_G \mathcal{O} \to T^*(Q/G)$ is given by $\tilde{\pi}_{\tilde{Q}}([(\alpha_{[q]}, q), \mu]) = \alpha_{[q]}$.

Denote the two form $\omega^{\mathcal{A}}$ in this situation by $\tilde{\omega}_{\mathcal{O}}^{-}$ and hence it is uniquely characterized by the relation

$$\rho^{*}\tilde{\omega}_{\mathcal{O}}^{-} = \mathbf{d}\left\langle \Pi_{\mathcal{O}}^{*}\mathbf{J}_{\mathcal{O}}, \Pi_{\tilde{Q}}^{*}\tilde{\mathcal{A}}\right\rangle + \Pi_{\mathcal{O}}^{*}\omega_{\mathcal{O}}^{-},$$

where $\Pi_{\tilde{Q}} : \tilde{Q} \times \mathcal{O} \to \tilde{Q}$ and $\Pi_{\mathcal{O}} : \tilde{Q} \times \mathcal{O} \to \mathcal{O}$ are the projections on the two factors.

The two-form $\tilde{\omega}_{\mathcal{O}}^{-} + \tilde{\pi}_{\tilde{Q}}^{*}\Omega_{Q/G}$ on $\tilde{Q} \times_G \mathcal{O}$ is obtained by reduction.

- Recall: The G-equivariant vector bundle isomorphism $\varphi_{\mathcal{A}} : \tilde{Q} \times \mathfrak{g}^{*} \to T^{*}Q$ is defined by $\varphi_{\mathcal{A}}\left(\left(\alpha_{[q]}, q\right), \mu\right) := T_q^{*}\pi(\alpha_{[q]}) + \mathcal{A}(q)^{*}\mu$ for any $\left(\left(\alpha_{[q]}, q\right), \mu\right) \in \tilde{Q} \times \mathfrak{g}^{*}$.

• Let $\mathbf{J}_{T^*Q} : T^*Q \to \mathfrak{g}^*$ be the momentum map of the lifted G-action. Define $\mathbf{J}_{\mathcal{A}} := \mathbf{J}_{T^*Q} \circ \varphi_{\mathcal{A}} : \tilde{Q} \times \mathfrak{g}^* \to \mathfrak{g}^*$. Then $\mathbf{J}_{\mathcal{A}} = \Pi_{\mathfrak{g}^*}$, the projection on the second factor. Hence $\mathbf{J}_{\mathcal{A}}^{-1}(\mathcal{O}) = \tilde{Q} \times \mathcal{O}$.

• $\Omega_{\mathcal{A}} = -\mathbf{d}\Theta_{\mathcal{A}}$ is a symplectic form on $\tilde{Q} \times \mathfrak{g}^*$, where

$$\begin{aligned}\Theta_{\mathcal{A}}\left(\left(\alpha_{[q]}, q\right), \mu\right)&\left(\left(u_{\alpha_{[q]}}, v_q\right), \nu\right) \\ &= \left\langle \alpha_{[q]}, T_q\pi(v_q)\right\rangle + \langle \mu, \mathcal{A}(q)(v_q)\rangle\end{aligned}$$

$\left(\left(\alpha_{[q]}, q\right), \mu\right) \in \tilde{Q} \times \mathfrak{g}^*$, $\left(u_{\alpha_{[q]}}, v_q\right) \in T_{\left(\alpha_{[q]}, q\right)}\tilde{Q}$, $\nu \in \mathfrak{g}^*$.

• So $\mathbf{J}_{\mathcal{A}} : \tilde{Q} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the equivariant momentum map of the canonical G-action on the symplectic manifold $\left(\tilde{Q} \times \mathfrak{g}^*, \Omega_{\mathcal{A}}\right)$.

- Therefore, $\tilde{Q} \times_G \mathcal{O} = \mathbf{J}_{\mathcal{A}}^{-1}(\mathcal{O})/G$ has the reduced symplectic form $\tilde{\omega}_{\mathcal{O}}^{-} + \tilde{\pi}_{\tilde{Q}}^{*}\Omega_{Q/G}$.

The symplectic leaves of S are the connected components of the symplectic manifolds $\left(\tilde{Q} \times_G \mathcal{O}, \tilde{\omega}_{\mathcal{O}}^{-} + \tilde{\pi}_{\tilde{Q}}^{*}\Omega_{Q/G}\right)$, where $\mathcal{O}$ is a coadjoint orbit in $\mathfrak{g}^*$.

Symplectic leaves of W

Recall that $\Phi : S \to W$ given by

$$\Phi\left(\left[(\alpha_{[q]}, q), \mu\right]\right) = \left(\alpha_{[q]}, [q, \mu]\right)$$

is a Poisson diffeomorphism. Therefore, the symplectic leaves of the Poisson manifold $(W, \{\,,\}_W)$ are the connected components of the symplectic manifolds

$$\left(\Phi\left(\tilde{Q} \times_G \mathcal{O}\right), \Phi_*\left(\tilde{\omega}_{\mathcal{O}}^- + \tilde{\pi}_{\tilde{Q}}^* \Omega_{Q/G}\right)\right).$$

Who are they?

$$
\begin{aligned}
\Phi & \left(\tilde{Q} \times_G \mathcal{O}\right) \\
& = \left\{\left(\alpha_{[q]}, [q, \mu]\right) \mid q \in Q, \alpha_{[q]} \in T_{[q]}(Q/G), \mu \in \mathcal{O} \subset \mathfrak{g}^*\right\} \\
& = T^*(Q/G) \oplus (Q \times_G \mathcal{O}) \\
& \subset W = T^*(Q/G) \oplus \tilde{\mathfrak{g}}^* = T^*(Q/G) \oplus (Q \times_G \mathfrak{g}^*).
\end{aligned}
$$

Here, $T^*(Q/G) \oplus (Q \times_G \mathcal{O})$ is a fiber subbundle, not a vector subbundle, of $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$; we still use the Whitney sum symbol, even though it is a fibered product of fiber bundles, to recall the fact that it is a subbundle of the Whitney sum bundle $W = T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$.

The closed G-invariant two-form $\omega^{-}_{Q\times\mathcal{O}} \in \Omega^2(Q\times\mathcal{O})$ defined by

$$\begin{aligned}\omega^{-}_{Q\times\mathcal{O}}(q,\mu)&\left((u_q,-\operatorname{ad}^*_\xi\mu),(v_q,-\operatorname{ad}^*_\eta\mu)\right)\\ &:= -\mathbf{d}(\mathcal{A}\times \mathrm{id}_{\mathcal{O}})(q,\mu)\left((u_q,-\operatorname{ad}^*_\xi\mu),(v_q,-\operatorname{ad}^*_\eta\mu)\right)\\ &\qquad + \omega^{-}_{\mathcal{O}}(\mu)\left(-\operatorname{ad}^*_\xi\mu,-\operatorname{ad}^*_\eta\mu\right),\end{aligned}$$

where $\mathcal{A}\times \mathrm{id}_{\mathcal{O}} \in \Omega^1(Q\times\mathfrak{g}^*)$ is given by

$$(\mathcal{A}\times \mathrm{id}_{\mathcal{O}})(q,\mu)\left(u_q,-\operatorname{ad}^*_\xi\mu\right) = \langle \mu, \mathcal{A}(q)(u_q)\rangle,$$

drops to a closed two-form $\omega^{-}_{Q\times_G\mathcal{O}} \in \Omega^2(Q\times_G\mathcal{O})$, that is, $\omega^{-}_{Q\times_G\mathcal{O}}$ is uniquely determined by the identity

$$\pi^*_{Q\times\mathcal{O}}\omega^{-}_{Q\times_G\mathcal{O}} = \omega^{-}_{Q\times\mathcal{O}},$$

where $\pi_{Q\times\mathcal{O}} : Q\times\mathcal{O}\to Q\times_G\mathcal{O}$ the orbit space projection.

The symplectic leaves of W are the connected components of the symplectic manifolds

$$\left(T^*(Q/G) \oplus (Q \times_G \mathcal{O}), \Pi^*_{T^*(Q/G)}\Omega_{Q/G} + \Pi^*_{Q\times_G\mathcal{O}}\omega^-_{Q\times_G\mathcal{O}}\right),$$

where $\mathcal{O}$ is a coadjoint orbit in $\mathfrak{g}^*$, $\Omega_{Q/G}$ is the canonical symplectic form on $T^*(Q/G)$, $\omega^-_{Q\times_G\mathcal{O}}$ is the closed two-form on $Q \times_G \mathcal{O}$ given above, and $\Pi_{T^*(Q/G)} : T^*(Q/G) \oplus (Q \times_G \mathcal{O}) \to T^*(Q/G)$, $\Pi_{Q\times_G\mathcal{O}} : T^*(Q/G) \oplus (Q \times_G \mathcal{O}) \to Q \times_G \mathcal{O}$ are the projections on the two factors.

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