# NEMATODYNAMICS

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**IMPA, Mathematical Physics Seminar, November 17, 2015**

# PLAN OF THE PRESENTATION

**Liquid crystal dynamics: two models**

**Euler-Poincaré reduction**

**Affine Euler-Poincaré reduction**

**Euler-Poincaré formulation for liquid crystal dynamics**

**Eringen implies Ericksen-Leslie**

**Analytical results**

All models are conservative: the terms modeling dissipation have been eliminated. The reason is that we want to understand the geometric nature of these equations. The dissipative terms can be added later. Think: Euler versus Navier-Stokes.

# LIQUID CRYSTALS

**Tutorial:** http://dept.kent.edu/spie/liquidcrystals/ by B. Senyuk

**Liquid crystal:** state of matter between crystal and isotropic liquid. Liquid behavior: high fluidity, formation and coalescence of droplets. Crystal behavior: anisotropy in optical, mechanical, electrical, magnetic properties. Long-range orientational order in their molecules and sometimes translational or positional order. Many phases; differ by their structure and physical properties.

**History:** Friedrich Reinitzer (1857 Prague – 1927 Graz) botanist and chemist. In 1888 he accidentaly discovered a strange behavior of cholesteryl benzoate that would later be called “liquid crystal". Work continued by physicist Otto Lehmann (1855 Konstanz – 1922 Karlsruhe), the “father" of liquid crystal technology. The discovery received plenty of attention at the time but due to no practical use the interest dropped soon. Lehmann was nominated 10 times (1913-22) for the Nobel Prize; never got it.

Georges Friedel (1865 Mulhouse – 1933 Strasbourg) mineralogist and crystallographer; in 1922 first classification of liquid crystals.

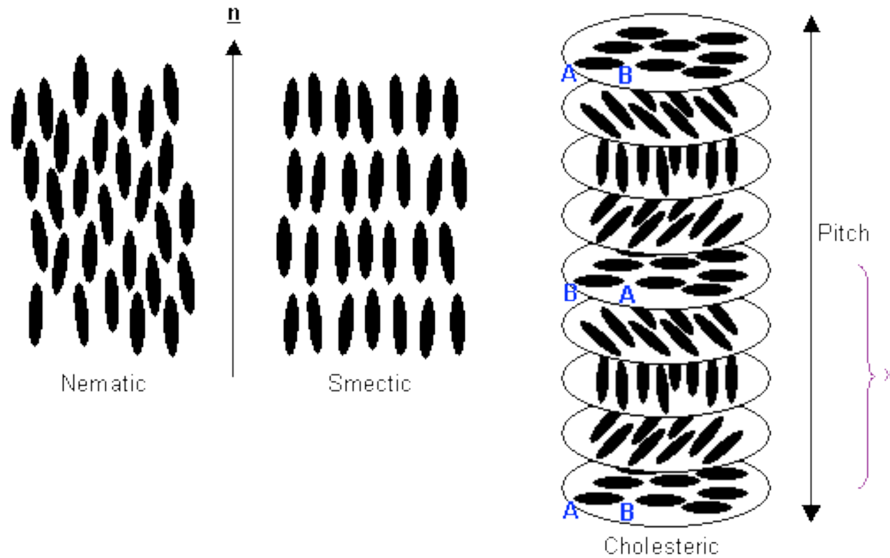
Carl Oseen (1879 Lund – 1944 Uppsala) theoretical physicist; first formulation of elasticity theory of liquid crystals.

Since the mid 1960s the entire theoretical and experimental development in liquid crystals has been influenced by the physicist Pierre-Gilles de Gennes (1932 Paris – 2007 Orsay) who got the Nobel Prize in 1991. Unfortunately, his book is not very useful to mathematicians. Better books are by Subrahmanyan Chandrasekhar and especially Epifanio Virga, the most mathematical book I came across.

Mathematical formulation was driven by engineers by posing the questions to applied mathematicians: Ericksen, Leslie, Lhuillier, Rey, Eringen.

Main phases of liquid crystals are the **nematic**, **smectic**, **cholesteric** (**chiral nematic**); molecules behave differently.

nematic $\sim$ “nema" (thread); smectic $\sim$ “smektos"(smeared), or “smechein" (to wash out) + “tikos" (suffix for adjectives of Greek origin); cholesteric $\sim$ “khole"(bile) + “steros"(solid, stiff) + “tikos"; chiral $\sim$ “cheir"(hand), introduced by Kelvin.




From: DoITPoMS, University of Cambridge

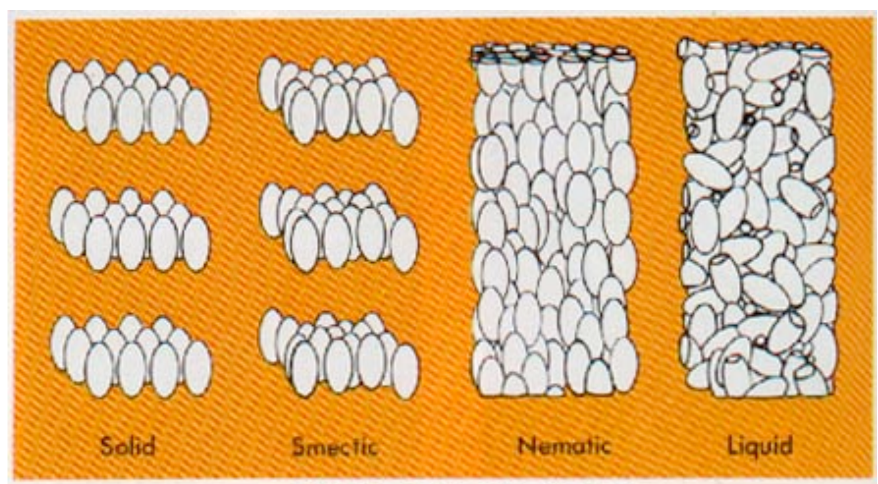
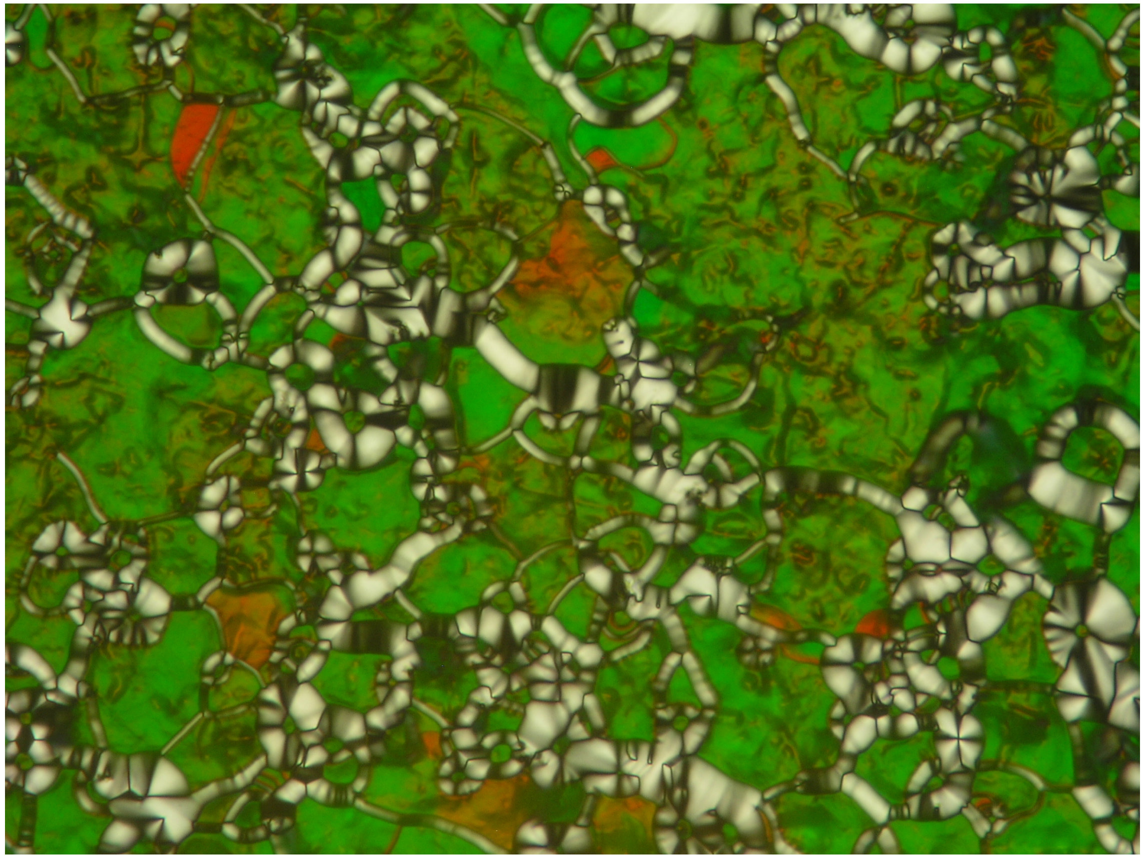
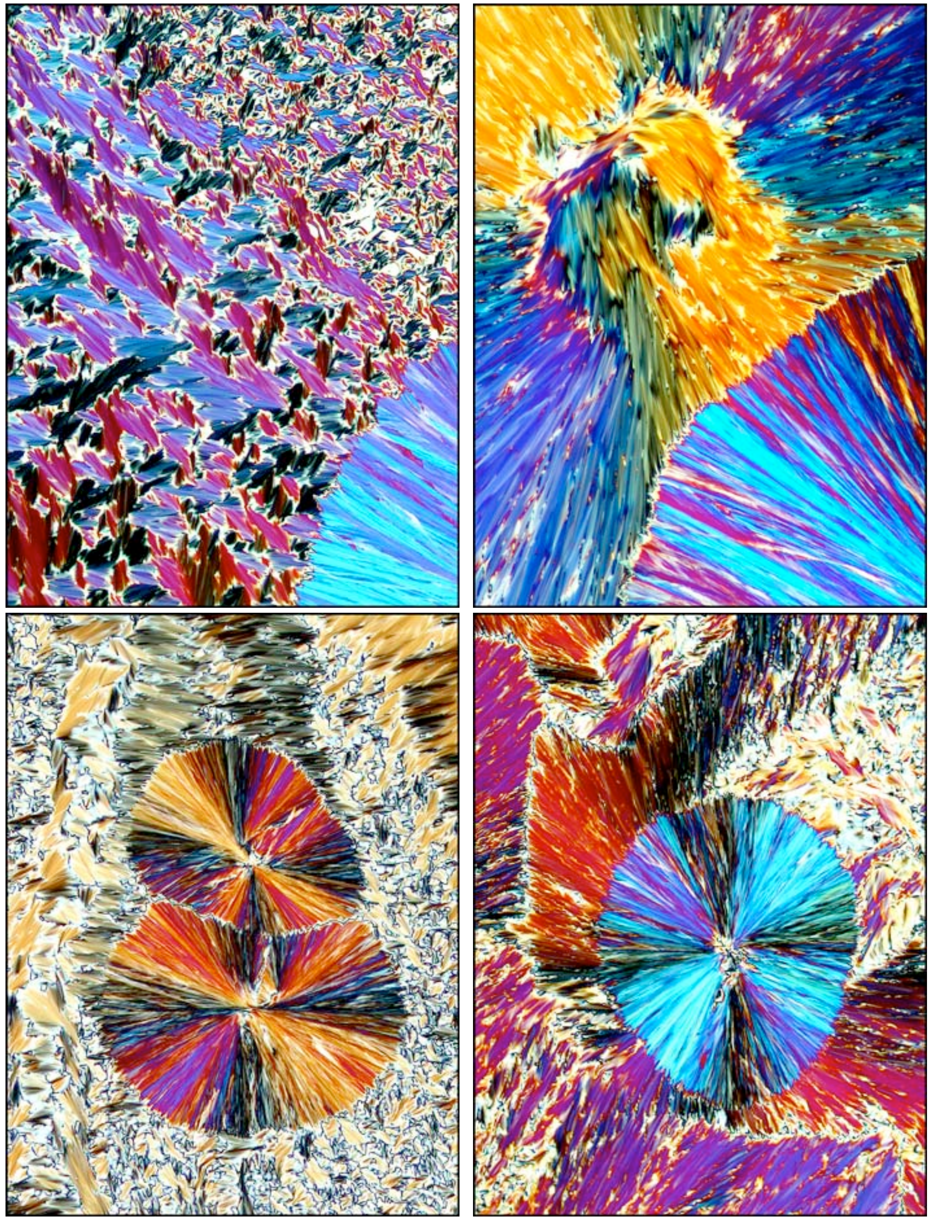



FIGURE 16-18. This diagram shows the differences in the degree of order for a solid, liquid crystal, and liquid.

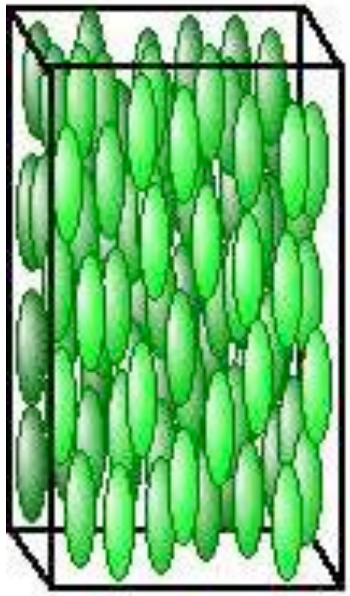
From: http://boomeria.org/chemtextbook/cch16.html

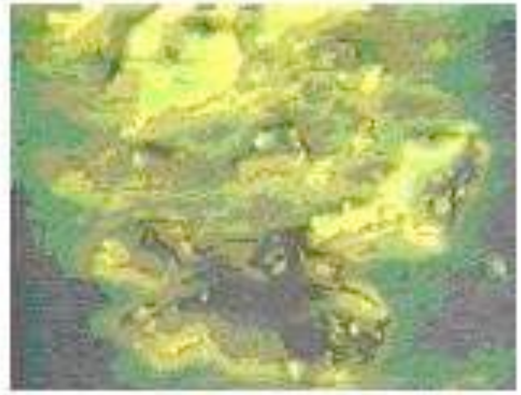


A microscope image showing that a solution of tiny DNA molecules has formed a liquid-crystal phase. The DNA molecules pair to form DNA double helices, which, in turn, stack end-to-end to make rod-shaped aggregates that orient parallel to one another. Image by Michi Nakata, University of Colorado.



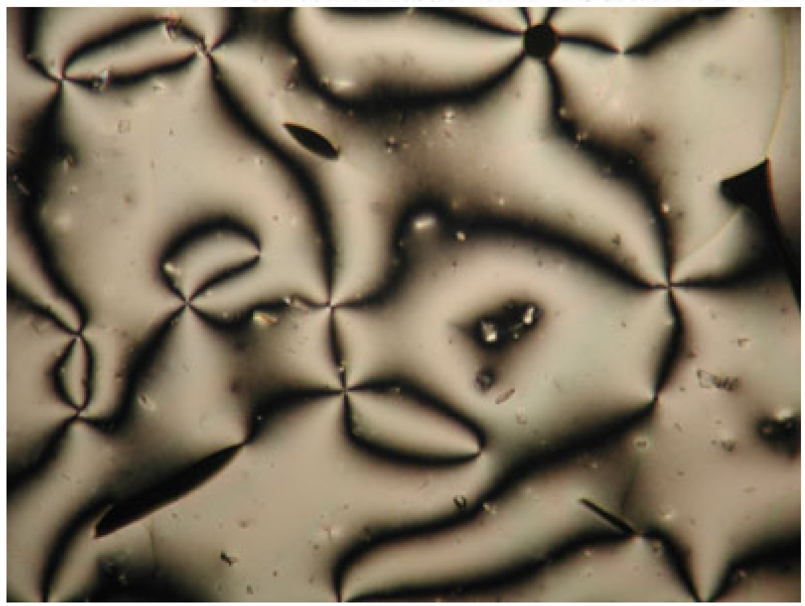
Photos by Brian Johnstone. Cooling from liquid crystal state to solid crystal state. The circular discs are the solid crystals.



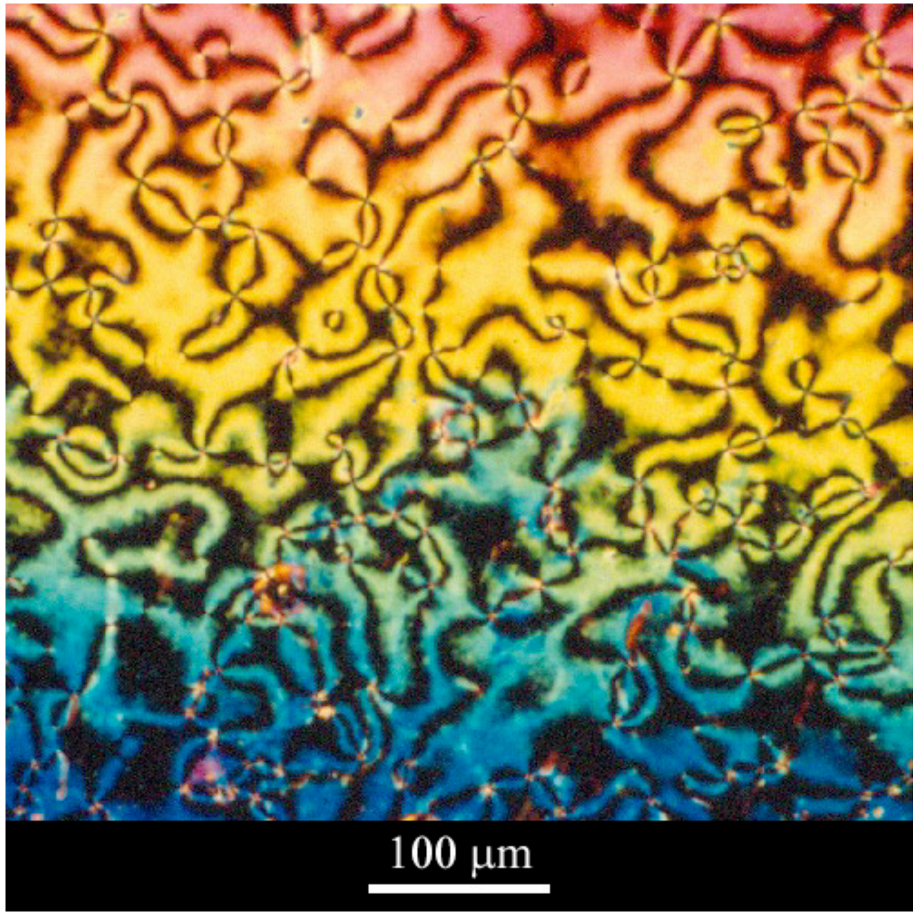


A schematic representation of the nematic phase (left) and a photo of a nematic liquid crystal (above).

*Photo courtesy Dr. Mary Neubert LCI-KSU*



Nematic liquid crystal. Note disclinations.
From: DoITPoMS, University of Cambridge




Nematic polymer; from: DoITPoMS, University of Cambridge

The thick, slippery substance found at the bottom of a soap dish is a type of smectic liquid crystal.

Molecules in this phase have translational order not present in the nematic phase. The molecules maintain orientational and align themselves in layers. Motion is restricted to within these planes and separate planes flow past each other. Increased order means that the smectic state is more “solid-like" than the nematic.

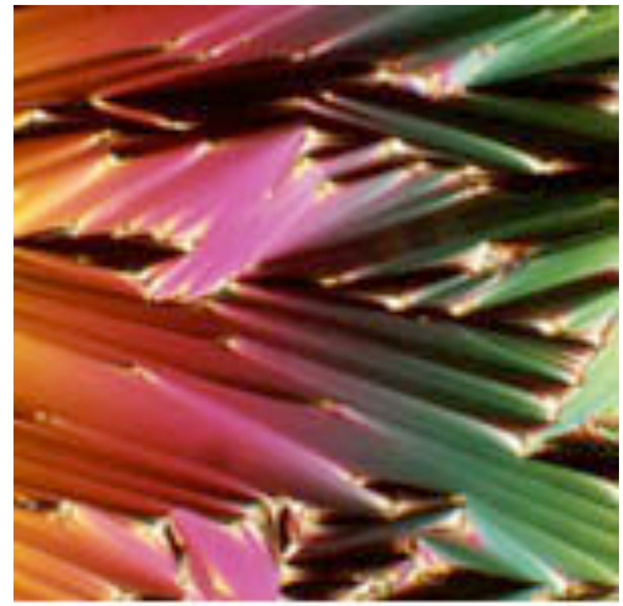


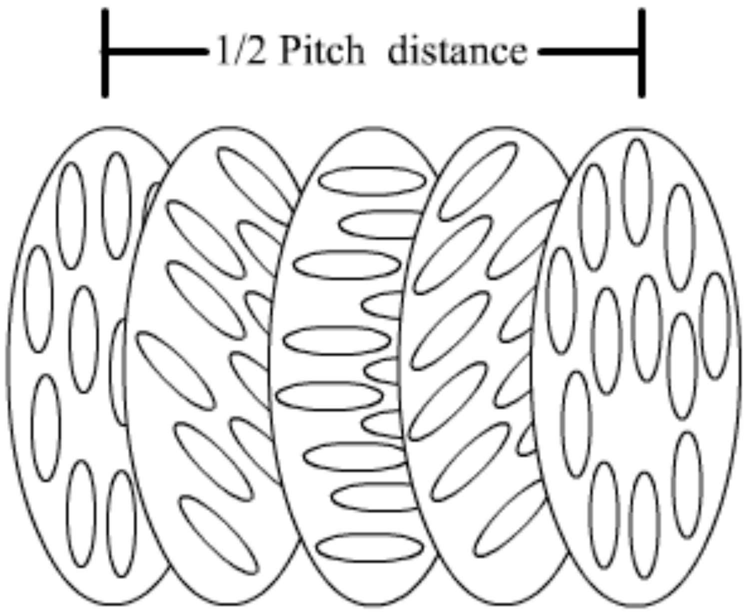
*Photo courtesy of Dr. Mary Neubert LCI-KSU*

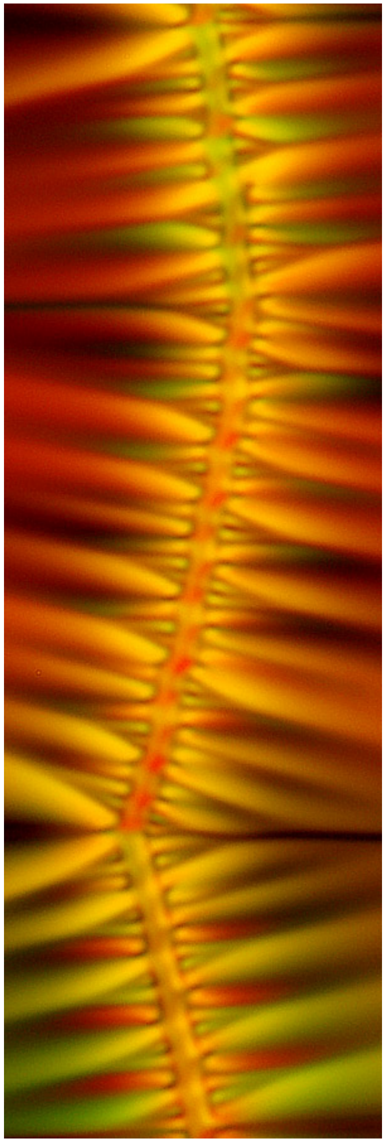
Smectic-A mesophase: the director is perpendicular to the smectic plane, no particular positional order in the layer.

Smectic-B mesophase: the director is perpendicular to the smectic plane, but the molecules are arranged into a network of hexagons within the layer (no picture).

Photo courtesy Dr. Mary Neubert LCI-KSU

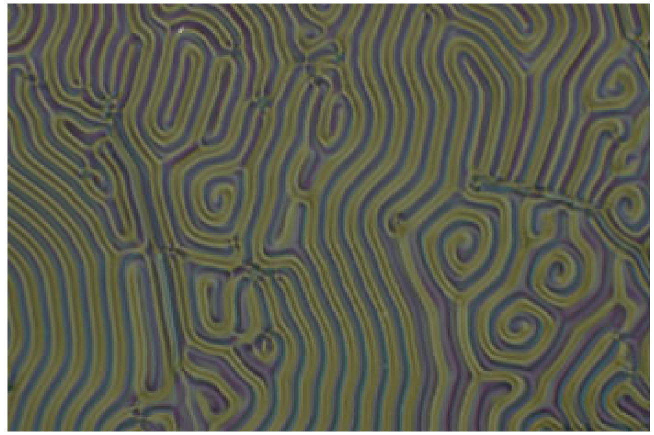
In the smectic-C mesophase, molecules are arranged as in the smectic-A mesophase, but the director is at a constant tilt angle relative to the smectic plane.





From B. Senyuk at http://dept.kent.edu/spie/liquidcrystals/

Cholesteric liquid crystal



M. Mitov/CNRS Photothèque. Cholesteric liquid crystal

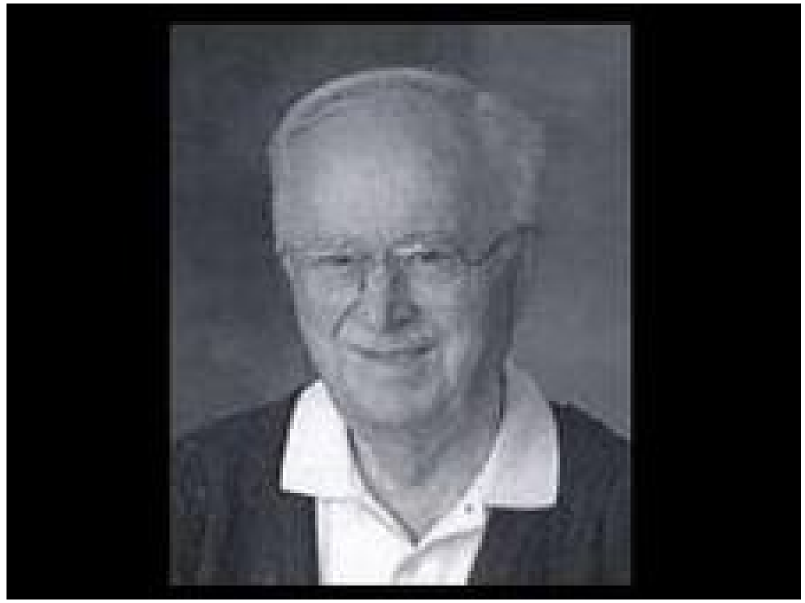
CNRS-CEMES 2006: Iridescent colors of this rose chafer beetle are due to the organization in cholesteric liquid crystal phase of the chitin molecules of the outer part of the shell

Jerry Ericksen, born December 20, 1924.

Photo during an interview with John Ball on May 28, 2013

Leslie and Ericksen in Glasgow in the mid 1970s

Frank Matthews Leslie (March 8, 1935 – June 15, 2000)



A. Cemal Eringen (February 15, 1921 – December 7, 2009)

# LIQUID CRYSTAL DYNAMICS

**Director theory** due to Oseen, Frank, Zöcher, Ericksen and Leslie

**Micropolar** and **microstretch theories**, due to Eringen, which take into account the microinertia of the particles and which is applicable, for example, to *liquid crystal polymers*

**Ordered micropolar** approach, due to Lhuillier and Rey, which combines the director theory with the micropolar models.

We discuss only nematic liquid crystals ($K_1 = 0$ in free energy; no chirality $\mathbf{n} \cdot \operatorname{curl} \mathbf{n}$). We set all dissipation equal to zero; want to understand the conservative case first.

$\mathcal{D} \subset \mathbb{R}^3$ bounded domain with smooth boundary. All boundary conditions are ignored: in all integration by parts the boundary terms vanish. We fix a volume form $\mu$ on $\mathcal{D}$.

## ERICKSEN-LESLIE DIRECTOR THEORY

### For nematic and cholesteric liquid crystals

**Key assumption**: only the direction and not the sense of the molecules matter. The preferred orientation of the molecules around a point is described by a unit vector $\mathbf{n} : \mathcal{D} \to S^2$, called the *director*, and $\mathbf{n}$ and $-\mathbf{n}$ are assumed to be equivalent.

**Ericksen-Leslie equations** (*Ericksen [1966], Leslie [1968]*) in a domain $\mathcal{D}$, constraint $\|\mathbf{n}\| = 1$, are:

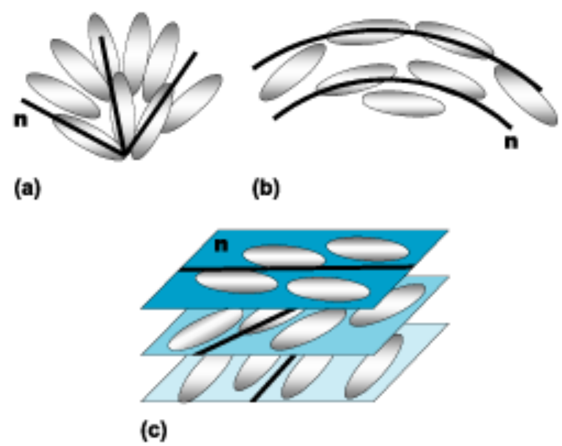
$$\begin{cases} \rho \left( \dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} \right) = \operatorname{grad} \dfrac{\partial F}{\partial \rho^{-1}} - \partial_j \left( \rho \dfrac{\partial F}{\partial \mathbf{n}_{,j}} \cdot \nabla \mathbf{n} \right), \\ \rho J \dfrac{D^2}{Dt^2}\mathbf{n} - 2q\mathbf{n} + \mathbf{h} = 0, \quad \mathbf{h} = \rho \dfrac{\partial F}{\partial \mathbf{n}} - \partial_i \left( \rho \dfrac{\partial F}{\partial \mathbf{n}_{,i}} \right) \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho \mathbf{u}) = 0, \qquad \dfrac{D}{Dt} := \dfrac{\partial}{\partial t} + \nabla_{\mathbf{u}} = \dfrac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \end{cases}$$

$\mathbf{u}$ *Eulerian velocity*, $\rho$ *mass density*, $\mathbf{n} : \mathcal{D} \to \mathbb{R}^3$ *director* ($\mathbf{n}$ equivalent to -$\mathbf{n}$), $J$ *microinertia constant*, and $F(\mathbf{n}, \mathbf{n}_{,i})$ is the *free energy*:

A standard choice for $F$ is the *Oseen-Zöcher-Frank free energy*:

$$\rho F(\rho^{-1}, \mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}K_{11}\underbrace{(\operatorname{div}\mathbf{n})^2}_{\textbf{splay}} + \frac{1}{2}K_{22}\underbrace{(\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2}_{\textbf{twist}}$$
$$+\frac{1}{2}K_{33}\underbrace{\|\mathbf{n}\times\operatorname{curl}\mathbf{n}\|^2}_{\textbf{bend}},$$

associated to the basic type of director distorsions nematics:




(a) splay, (b) bend, (c) twist

WHAT IS THE VARIATIONAL/HAMILTONIAN STRUCTURE OF THESE EQUATIONS?

# ERINGEN MICROPOLAR THEORY

**First key assumption**: Replace point particles by *small deformable bodies*: **microfluids**. Examples: *liquid crystals, blood, polymer melts, bubbly fluids, suspensions with deformable particles, biological fluids*. *Eringen [1978], [1979], [1981],...*

A material particle $P$ in a microfluid is characterized by its position $X$ and by a vector $\Xi$ attached to $P$ that denotes the orientation and intrinsic deformation of $P$. Both $X$ and $\Xi$ have their own motions: $X \mapsto x = \eta(X,t)$ and $\Xi \mapsto \xi = \chi(X,\Xi,t)$ called, respectively, the *macromotion* and *micromotion*.

**Second key assumption**: Material bodies are very small, so a linear approximation in $\Xi$ is permissible for the micromotion:

$$\xi = \chi(X,t)\Xi,$$

where $\chi(X,t) \in \mathrm{GL}(3)^+ := \{A \in \mathrm{GL}(3) \mid \det(A) > 0\}$.

The classical Eringen theory considers only three possible groups in the description of the micromotion of the particles:

$$\mathrm{GL}(3)^+(\textit{micromorphic}) \supset \mathrm{K}(3)(\textit{microstretch}) \supset \mathrm{SO}(3)(\textit{micropolar}),$$

$$\mathrm{K}(3) = \left\{ A \in \mathrm{GL}(3)^+ \mid \text{there exists } \lambda \in \mathbb{R} \text{ such that } AA^T = \lambda I_3 \right\}$$

is a closed subgroup of $\mathrm{GL}(3)^+$; associated to rotations and stretch.

The general theory admits other groups describing the micromotion.

We will study only micropolar fluids, i.e., the **order parameter group** is

$$\mathcal{O} := \mathrm{SO}(3)$$

.

Eringen's equations for non-dissipative micropolar liquid crystals:

$$\begin{cases} \rho \dfrac{D}{Dt}\mathbf{u}_l = \partial_l \dfrac{\partial \Psi}{\partial \rho^{-1}} - \partial_k \left( \rho \dfrac{\partial \Psi}{\partial \gamma^a_k} \gamma^a_l \right), \quad \rho \sigma_l = \partial_k \left( \rho \dfrac{\partial \Psi}{\partial \gamma^l_k} \right) - \varepsilon_{lmn} \rho \dfrac{\partial \Psi}{\partial \gamma^a_m} \gamma^a_n, \\ \dfrac{D}{Dt}\rho + \rho \operatorname{div} \mathbf{u} = 0, \qquad \dfrac{D}{Dt} j_{kl} + (\varepsilon_{kpr} j_{lp} + \varepsilon_{lpr} j_{kp}) \boldsymbol{\nu}_r = 0, \\ \dfrac{D}{Dt}\gamma^a_l = \partial_l \boldsymbol{\nu}_a + \nu_{ab} \gamma^b_l - \gamma^a_r \partial_l u_r, \qquad \dfrac{D}{Dt} := \dfrac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \quad \text{mat. deriv.} \end{cases}$$

sum on repeated indices, $\mathbf{u} \in \mathfrak{X}(\mathcal{D})$ Eulerian velocity, $\rho \in \mathcal{F}(\mathcal{D})$ mass density, $\boldsymbol{\nu} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3)$ microrotation rate, where we use the standard isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$, $j_{kl} \in \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3))$ microinertia tensor (symmetric), $\sigma_k$ spin inertia is defined by

$$\sigma_k := j_{kl} \frac{D}{Dt} \boldsymbol{\nu}_l + \varepsilon_{klm} j_{mn} \boldsymbol{\nu}_l \boldsymbol{\nu}_n = \frac{D}{Dt}(j_{kl} \boldsymbol{\nu}_l),$$

$\gamma = (\gamma^{ab}_i) \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$ wryness tensor, related to $(\eta, \chi)$ by

$$\gamma = -\eta_* (\nabla \chi) \chi^{-1} =: \widehat{\boldsymbol{\gamma}} = \widehat{(\boldsymbol{\gamma}^a_i)},$$

and $\Psi = \Psi(\rho^{-1}, j, \gamma) : \mathbb{R} \times \mathrm{Sym}(3) \times \mathfrak{gl}(3) \to \mathbb{R}$ is the free energy.

WHAT IS THE VARIATIONAL/HAMILTONIAN STRUCTURE OF THESE EQUATIONS?

WHAT IS THE RELATION BETWEEN ERICKSEN-LESLIE AND ERINGEN THEORY?

Eringen's claim: Eringen theory recovers Ericksen-Leslie theory in the rod-like assumption $j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$ with the choice $\gamma = \nabla \mathbf{n} \times \mathbf{n}$.

Once we have the Euler-Poincaré formulation, it will be clear that $\gamma = \nabla \mathbf{n} \times \mathbf{n}$ cannot be considered as a definition! This is FALSE!

This statement has been controversial due to mistakes:

e.g. paper by Rymarz [1990]

*Int. J. Engng Sci.* Vol. 28, No. 1, pp. 11–21, 1990
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# MORE ABOUT THE RELATIONS BETWEEN THE ERICKSEN–LESLIE–PARODI AND ERINGEN–LEE THEORIES OF NEMATIC LIQUID CRYSTALS

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**Abstract**—This paper is a further consideration of the relation between two main phenomenological theories of nematic liquid crystals: Ericksen–Leslie–Parodi (ELP) and Eringen–Lee (EL). The aim of the study is to establish the generality of the conclusion which claims that the ELP theory is a particular case of the EL theory.

According to the analysis presented in the paper this conclusion may be treated as true but only after modification of the constitutive equation for ${}_Em_{ij}$ in the EL theory by one term arising from splay deformation. The results of the study are formulated in four conclusions given at the end of the paper.

This was soon reconsidered in Eringen [1993]

*Int. J. Engng Sci.* Vol. 31, No. 4, pp. 605–616, 1993
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# AN ASSESSMENT OF DIRECTOR AND MICROPOLAR THEORIES OF LIQUID CRYSTALS

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**Abstract**—It is shown that of the two prominent theories of liquid crystals, the Micropolar theory is more general than the Director theory. Under special assumptions, for liquid crystals having rigid rod-like molecular elements, the Micropolar theory reduces to the Director theory. The relationship between the two theories is established fully. An assessment is made of the limitations of both theories and on their domain of applicability.

more precisely

The present discussion is concerned with the relationship of OFEL and E-theories. Already discussions exist in this regard (e.g. [12] and [13]). Rymarz [12] has shown that OFEL-theory (he calls it ELP-theory) is a special case of E-theory, if the stress potential is modified by a splay term $K(\operatorname{div} \mathbf{n})^2$. Below I shall show that this splay term is already present in the E-theory. Consequently, his statement should be modified to: *The OFEL-theory is a special case of the E-theory.*

The major contributions of this article is not only in this correction, but in the display of relations between the two theories, critical examination of their physical foundations and domain of applicability of each theory in regard to liquid crystals that possess more complicated molecular structures. In particular, this paper should serve the following purposes:

This was an open problem for about 25 years.

We solve this problem using techniques of geometric mechanics:

(1) we show under which assumptions, Eringen reduces to Ericksen-Leslie

(2) we establish the correct relation between $\gamma$ and $\mathbf{n}$ under these assumptions.

$\rightsquigarrow$ Both Eringen and Rymarz are partially right!

# EULER-POINCARÉ REDUCTION

**Poincaré 1901:** Left (right) invariant Lagrangian $L : TG \to \mathbb{R}$, $l := L|_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{R}$.
For $g(t) \in G$, let $\xi(t) = g(t)^{-1}\dot{g}(t)$ $(\dot{g}(t)g(t)^{-1} \in \mathfrak{g})$.
Then the following are equivalent:

**(i)** $g(t)$ satisfies the Euler-Lagrange equations for $L$ on $G$.

**(ii)** The variational principle

$$\delta \int_a^b L(g(t), \dot{g}(t))dt = 0$$

holds, for variations with fixed endpoints.

**(iii)** The **Euler-Poincaré equations** hold: $\dfrac{d}{dt}\dfrac{\delta l}{\delta \xi} = \pm \mathrm{ad}^*_\xi \dfrac{\delta l}{\delta \xi}$.

**(iv)** The **Euler-Poincaré variational principle**

$$\delta \int_a^b l(\xi(t))dt = 0$$

holds on $\mathfrak{g}$, for variations $\delta\xi = \dot{\eta} \pm [\xi, \eta]$, where $\eta(t)$ is an arbitrary path in $\mathfrak{g}$ that vanishes at the endpoints, i.e $\eta(a) = \eta(b) = 0$.

Geometry has led to analytic questions. I am not aware of any serious analysis results for such constrained variational principles.

## Reconstruction

Solve the Euler-Lagrange equations for a left invariant $L : TG \to \mathbb{R}$

• Form $l := L|_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{R}$

• Solve the Euler-Poincaré equations: $\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}$ , $\xi(0) = \xi_0$

• Solve linear equation with time dependent coefficients (quadrature): $\dot{g}(t) = g(t)\xi(t)$, $g(0) = e$

• For any $g_0 \in G$ the solution of the Euler-Lagrange equations is $V(t) = g_0 g(t)\xi(t)$ with initial condition $V(0) = g_0\xi_0$.

EP reduction: free rigid body, ideal fluids, KdV

EP reduction for semidirect products: heavy rigid body, compressible fluids, MHD, GFD. *Holm, Marsden, Ratiu [1998]*

Geometry of complex fluids. *Holm [2002], Gay-Balmaz, Ratiu [2009]*

# AFFINE EULER-POINCARÉ REDUCTION

*Right* $G$-representation on $V$, $(v, g) \in V \times G \mapsto vg \in V$, induces:

- right $G$-representation on $V^*$: $(a, g) \in V^* \times G \mapsto ag \in V^*$

- right $\mathfrak{g}$-representation on $V$: $(v, \xi) \in V \times \mathfrak{g} \mapsto v\xi \in V$

- right $\mathfrak{g}$-representation on $V^*$: $(a, \xi) \in V^* \times \mathfrak{g} \mapsto a\xi \in V^*$

Duality pairings: $\langle\, , \rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle\, , \rangle_V : V^* \times V \to \mathbb{R}$

Affine *right* representation: $\theta_g(a) = ag + c(g)$, where $c \in \mathcal{F}(G, V^*)$ is a right group one-cocycle, i.e., $c(fg) = c(f)g + c(g)$, $\forall f, g \in G$. This implies that $c(e) = 0$ and $c(g^{-1}) = -c(g)g^{-1}$. Note that

$$\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(t\xi)}(a) = a\xi + \mathbf{d}c(\xi), \quad \xi \in \mathfrak{g}, \quad a \in V^*,$$

where $\mathbf{d}c : \mathfrak{g} \to V^*$ is defined by $\mathbf{d}c(\xi) := T_e c(\xi)$. Useful to introduce:

- $\mathbf{d}c^{\mathsf{T}} : V \to \mathfrak{g}^*$ by $\langle \mathbf{d}c^{\mathsf{T}}(v), \xi \rangle_{\mathfrak{g}} := \langle \mathbf{d}c(\xi), v \rangle_V$, for $\xi \in \mathfrak{g}$, $v \in V$

- $\diamond : V \times V^* \to \mathfrak{g}^*$ by $\langle v \diamond a, \xi \rangle_{\mathfrak{g}} := -\langle a\xi, v \rangle_V$ for $\xi \in \mathfrak{g}$, $v \in V$, $a \in V^*$

- then: $\langle a\xi + \mathbf{d}c(\xi), v \rangle_V = \langle \mathbf{d}c^{\mathsf{T}}(v) - v \diamond a, \xi \rangle_{\mathfrak{g}}$

- the semidirect product $S = G \circledS V$ with group multiplication

$$(g_1, v_1)(g_2, v_2) := (g_1 g_2, v_2 + v_1 g_2), \quad g_i \in G, \quad v_i \in V$$

- its Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$ with bracket

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) := [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1 \xi_2 - v_2 \xi_1)$$

- then for $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^* = \mathfrak{g}^* \times V^*$ we have

$$\mathrm{ad}^*_{(\xi, v)}(\mu, a) = (\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi)$$

**In a physical problem (like liquid crystals) we are given:**

• $L : TG \times V^* \to \mathbb{R}$ right $G$-invariant under the action
$(v_h, a) \in T_hG \times V^* \stackrel{g}{\longmapsto} (v_h g, \theta_g(a)) = (v_h g, ag + c(g)) \in T_{hg}G \times V^*$.

• So, if $a_0 \in V^*$, define $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then $L_{a_0}$ is right invariant under the lift to $TG$ of right translation of $G^c_{a_0}$ on $G$, where $G^c_{a_0}$ is the $\theta$-isotropy group of $a_0$.

• Right $G$-invariance of $L$ permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(v_g g^{-1}, \theta_{g^{-1}}(a_0)) = L(v_g, a_0).$$

• Curve $g(t) \in G$, let $\xi(t) := \dot{g}(t)g(t)^{-1} \in \mathfrak{g}$, $a(t) = \theta_{g(t)^{-1}}(a_0) \in V^*$ Then $a(t)$ as the unique solution of the following affine differential equation with time dependent coefficients

$$\dot{a}(t) = -a(t)\xi(t) - \mathbf{d}c(\xi(t)),$$

with initial condition $a(0) = a_0 \in V^*$.

The following are equivalent:

**(i)** With $a_0$ held fixed, Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

**(ii)** $g(t)$ satisfies the Euler-Lagrange equations for $L_{a_0}$ on $G$.

**(iii)** The constrained variational principle

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0,$$

holds on $\mathfrak{g} \times V^*$, upon using variations of the form

$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = -a\eta - \mathbf{d}c(\eta),$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

**(iv)** The affine Euler-Poincaré equations hold on $\mathfrak{g} \times V^*$:

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} = -\,\mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a - \mathbf{d}c^{\mathsf{T}}\left(\frac{\delta l}{\delta a}\right).$$

## Lagrangian Approach to Continuum Theories of Perfect Complex Fluids

To apply the previous theorem to complex fluids one makes two key observations:

**1.** Complex fluids have internal degrees of freedom encoded by the order parameter Lie group $\mathcal{O}$

**2.** New kind of advection equation: $\frac{D}{Dt}\gamma_l^a = \partial_l \boldsymbol{\nu}_a + \nu_{ab}\gamma_l^b - \gamma_r^a \partial_l u_r$

Geometrically, this means:

**1.** Enlarge the "particle relabeling group" $\mathrm{Diff}(\mathcal{D})$ to the semidirect product $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$, $\mathcal{F}(\mathcal{D}, \mathcal{O}) := \{\chi : \mathcal{D} \to \mathcal{O}$ smooth$\}$

**2.** The usual advection equations (for the mass density, the entropy, the magnetic field, etc) need to be augmented by a new advected quantity on which the group $G$ acts by an *affine representation*.

Algebraic structure of the symmetry group of complex fluids:

$\mathrm{Diff}(\mathcal{D})$ acts on $\mathcal{F}(\mathcal{D},\mathcal{O})$ via the *right* action

$$(\eta,\chi)\in \mathrm{Diff}(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathcal{O})\mapsto \chi\circ\eta\in\mathcal{F}(\mathcal{D},\mathcal{O}).$$

Therefore, the group multiplication is given by

$$(\eta,\chi)(\varphi,\psi)=(\eta\circ\varphi,(\chi\circ\varphi)\psi).$$

Fix a volume form $\mu$ on $\mathcal{D}$, so identify densities with functions, one-form densitities with one-forms, etc.

The Lie algebra $\mathfrak{g}$ of the semidirect product group is

$$\mathfrak{g}=\mathfrak{X}(\mathcal{D})\circledS\mathcal{F}(\mathcal{D},\mathfrak{o})\ni(\mathbf{u},\boldsymbol{\nu}),$$

and the Lie bracket is computed to be

$$\mathrm{ad}_{(\mathbf{u},\nu)}(\mathbf{v},\zeta)=(\mathrm{ad}_{\mathbf{u}}\mathbf{v},\mathrm{ad}_\nu\zeta+\mathbf{d}\nu\cdot\mathbf{v}-\mathbf{d}\zeta\cdot\mathbf{u}),$$

where $\mathrm{ad}_{\mathbf{u}}\mathbf{v}=-[\mathbf{u},\mathbf{v}]$, $\mathrm{ad}_\nu\zeta\in\mathcal{F}(\mathcal{D},\mathfrak{o})$ is given by $\mathrm{ad}_\nu\zeta(x):=\mathrm{ad}_{\nu(x)}\zeta(x)$, and $\mathbf{d}\nu\cdot\mathbf{v}\in\mathcal{F}(\mathcal{D},\mathfrak{o})$ is given by $\mathbf{d}\nu\cdot\mathbf{v}(x):=\mathbf{d}\nu(x)(\mathbf{v}(x))$.

The dual Lie algebra is identified with

$$\mathfrak{g}^* = \Omega^1(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \ni (\mathbf{m}, \kappa),$$

through the pairing

$$\langle (\mathbf{m}, \kappa), (\mathbf{u}, \nu) \rangle = \int_{\mathcal{D}} (\mathbf{m} \cdot \mathbf{u} + \kappa \cdot \nu)\, \mu.$$

The dual map to $\mathrm{ad}_{(\mathbf{u},\nu)}$ is

$$\mathrm{ad}^*_{(\mathbf{u},\nu)}(\mathbf{m}, \kappa) = \big(\pounds_{\mathbf{u}}\mathbf{m} + (\operatorname{div}\mathbf{u})\mathbf{m} + \kappa \cdot \mathbf{d}\nu, \mathrm{ad}^*_\nu \kappa + \operatorname{div}(\mathbf{u}\kappa)\big).$$

**Explanation of the symbols:**

- $\kappa \cdot \mathbf{d}\nu \in \Omega^1(\mathcal{D})$ denotes the one-form defined by

$$(\kappa \cdot \mathbf{d}\nu)(v_x) := \kappa(x)(\mathbf{d}\nu(v_x))$$

- $\mathrm{ad}^*_\nu \kappa \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ denotes the $\mathfrak{o}^*$-valued mapping defined by

$$\big(\mathrm{ad}^*_\nu \kappa\big)(x) := \mathrm{ad}^*_{\nu(x)}(\kappa(x)).$$

- $\mathbf{u}\kappa$ is the $1$-contravariant tensor field with values in $\mathfrak{o}^*$ defined by

$$(\mathbf{u}\kappa)(\alpha_x) := \alpha_x(\mathbf{u}(x))\kappa(x) \in \mathfrak{o}^*.$$

So $\mathbf{u}\kappa$ is a generalization of the notion of a vector field. $\mathfrak{X}(\mathcal{D}, \mathfrak{v}^*)$ denotes the space of all $\mathfrak{v}^*$-valued $1$-contravariant tensor fields.

• $\operatorname{div}(\mathbf{u})$ denotes the divergence of the vector field $\mathbf{u}$ with respect to the fixed volume form $\mu$. Recall that it is defined by the condition

$$(\operatorname{div}\mathbf{u})\mu = \pounds_{\mathbf{u}}\mu.$$

This operator can be naturally extended to the space $\mathfrak{X}(\mathcal{D}, \mathfrak{v}^*)$ as follows. For $w \in \mathfrak{X}(\mathcal{D}, \mathfrak{v}^*)$ we write $w = w_a \varepsilon^a$ where $(\varepsilon^a)$ is a basis of $\mathfrak{v}^*$ and $w_a \in \mathfrak{X}(\mathcal{D})$. We define $\operatorname{div} : \mathfrak{X}(\mathcal{D}, \mathfrak{v}^*) \to \mathcal{F}(\mathcal{D}, \mathfrak{v}^*)$ by

$$\operatorname{div} w := (\operatorname{div} w_a)\varepsilon^a.$$

Note that if $w = \mathbf{u}\kappa$ we have

$$\operatorname{div}(\mathbf{u}\kappa) = \mathbf{d}\kappa \cdot \mathbf{u} + (\operatorname{div}\mathbf{u})\kappa.$$

Split the space of advected quantities in two: usual ones and new ones that involve affine actions and cocycles.

## GEOMETRY OF THE ERICKSEN-LESLIE EQUATIONS

• Symmetry group: $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, SO(3)) \ni (\eta, \chi)$, macromotion and micromotion.

• Advected variables: $V^* = \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathbb{R}^3) \ni (\rho, \mathbf{n})$, mass density and director field.

• Representation of $G$ on $V^*$:

$$(\rho, \mathbf{n}) \mapsto \left( J(\eta)(\rho \circ \eta), \chi^{-1}(\mathbf{n} \circ \eta) \right).$$

• Associated infinitesimal actions and diamond operations:

$$\mathbf{n}\mathbf{u} = \nabla \mathbf{n} \cdot \mathbf{u}, \quad \mathbf{n}\boldsymbol{\nu} = \mathbf{n} \times \boldsymbol{\nu}, \quad \mathbf{m} \diamond_1 \mathbf{n} = -\nabla \mathbf{n}^{\mathsf{T}} \cdot \mathbf{m} \quad \text{and} \quad \mathbf{m} \diamond_2 \mathbf{n} = \mathbf{n} \times \mathbf{m},$$

where $\boldsymbol{\nu}, \mathbf{m}, \mathbf{n} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3)$.

• No cocycle.

- EP equations for $\left(\mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, SO(3))\right) \circledS \left(\mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathbb{R}^3)\right)$:

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}\dfrac{\delta \ell}{\delta \mathbf{u}} = -\pounds_{\mathbf{u}}\dfrac{\delta \ell}{\delta \mathbf{u}} - \operatorname{div} \mathbf{u}\dfrac{\delta \ell}{\delta \mathbf{u}} - \dfrac{\delta l}{\delta \boldsymbol{\nu}}\cdot \mathbf{d}\boldsymbol{\nu} + \rho\, \mathbf{d}\dfrac{\delta \ell}{\delta \rho} - \left(\nabla \mathbf{n}^{\mathsf{T}} \cdot \dfrac{\delta \ell}{\delta \mathbf{n}}\right)^{\flat}, \\ \dfrac{\partial}{\partial t}\dfrac{\delta \ell}{\delta \boldsymbol{\nu}} = \boldsymbol{\nu} \times \dfrac{\delta \ell}{\delta \boldsymbol{\nu}} - \operatorname{div}\left(\dfrac{\delta \ell}{\delta \boldsymbol{\nu}}\mathbf{u}\right) + \mathbf{n} \times \dfrac{\delta \ell}{\delta \mathbf{n}}, \end{array}\right.$$

- The advection equations are:

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \dfrac{\partial}{\partial t}\mathbf{n} + \nabla \mathbf{n} \cdot \mathbf{u} + \mathbf{n} \times \boldsymbol{\nu} = 0. \end{array}\right.$$

- Reduced Lagrangian for nematic and cholesteric liquid crystals:

$$\ell(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n}) := \frac{1}{2}\int_{\mathcal{D}} \rho \|\mathbf{u}\|^2 \mu + \frac{1}{2}\int_{\mathcal{D}} \rho J \|\boldsymbol{\nu}\|^2 \mu - \int_{\mathcal{D}} \rho F(\rho^{-1}, \mathbf{n}, \nabla \mathbf{n})\mu.$$

• EP equations for this $\ell$: yield

$$(motion) \quad \begin{cases} \rho\left(\frac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u}\right) = \operatorname{grad}\frac{\partial F}{\partial \rho^{-1}} - \partial_i\left(\rho\frac{\partial F}{\partial \mathbf{n}_{,i}}\cdot\nabla\mathbf{n}\right), \\ \rho J\frac{D}{Dt}\boldsymbol{\nu} = \mathbf{h}\times\mathbf{n}, \end{cases}$$

$$(advection) \quad \begin{cases} \frac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \\ \frac{D}{Dt}\mathbf{n} = \boldsymbol{\nu}\times\mathbf{n}, \end{cases}$$

• Recovering the Ericksen-Leslie equations:

Observation: if $\boldsymbol{\nu}$ and $\mathbf{n}$ are solutions of the EP equations then:

**(i)** $\|\mathbf{n}_0\| = 1$ *implies* $\|\mathbf{n}\| = 1$ *for all time.*

**(ii)** $\frac{D}{Dt}(\mathbf{n}\cdot\boldsymbol{\nu}) = 0$. *Therefore,* $\mathbf{n}_0\cdot\boldsymbol{\nu}_0 = 0$ *implies* $\mathbf{n}\cdot\boldsymbol{\nu} = 0$ *for all time.*

**(iii)** *Suppose that* $\mathbf{n}_0 \cdot \boldsymbol{\nu}_0 = 0$ *and* $\|\mathbf{n}_0\| = 1$. *Then*

$$\frac{D}{Dt}\mathbf{n} = \boldsymbol{\nu} \times \mathbf{n} \quad \textit{becomes} \quad \boldsymbol{\nu} = \mathbf{n} \times \frac{D}{Dt}\mathbf{n}$$

*and*

$$\rho J \frac{D}{Dt}\boldsymbol{\nu} = \mathbf{h} \times \mathbf{n} \quad \textit{becomes} \quad \rho J \frac{D^2}{Dt^2}\mathbf{n} - 2q\mathbf{n} + \mathbf{h} = 0.$$

Therefore:

*If* $(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n})$ *is a solution of the Euler-Poincaré equations with initial conditions* $\mathbf{n}_0$ *and* $\boldsymbol{\nu}_0$ *satisfying* $\|\mathbf{n}_0\| = 1$ *and* $\mathbf{n}_0 \cdot \boldsymbol{\nu}_0 = 0$, *then* $(\mathbf{u}, \rho, \mathbf{n})$ *is a solution of the Ericksen-Leslie equations.*

Conversely:

if $(\mathbf{u}, \rho, \mathbf{n})$ is a solution of Ericksen-Leslie equations, define

$$\boldsymbol{\nu} := \mathbf{n} \times \frac{D}{Dt}\mathbf{n} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3).$$

Then, $(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n})$ is a solution of the Euler-Poincaré equations.

# GEOMETRY OF THE ERINGEN EQUATIONS

• Symmetry group: same group as before $G = \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$.

• Advected variables: $V^* = \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \times \Omega^1(\mathcal{D}, \mathfrak{so}(3)) \ni (\rho, j, \gamma)$, mass density, microinertia tensor, strain.

• Representation: $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ acts *linearly* on the advected quantities $(\rho, j) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3))$, by

$$(\rho, j) \mapsto \left( J(\eta)(\rho \circ \eta), \chi^{\mathsf{T}}(j \circ \eta)\chi \right), \quad \chi^{\mathsf{T}} = \chi^{-1}.$$

• Affine representation: $(\eta, \chi) \in \mathrm{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$ acts on $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$ by an *affine* representation

$$\gamma \mapsto \chi^{-1}(\eta^* \gamma)\chi + \chi^{-1}\nabla\chi.$$

Note that $\gamma$ transforms as a connection.

• The reduced Lagrangian of Eringen's theory:

$$\ell : \left[\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathbb{R}^3)\right] \circledS \left[\mathcal{F}(\mathcal{D}) \oplus \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \oplus \Omega^1(\mathcal{D}, \mathfrak{so}(3))\right] \to \mathbb{R}$$

$$\ell(\mathbf{u}, \boldsymbol{\nu}, \rho, j, \gamma) = \frac{1}{2}\int_{\mathcal{D}} \rho \|\mathbf{u}\|^2 \mu + \frac{1}{2}\int_{\mathcal{D}} \rho\,(j\boldsymbol{\nu}\cdot\boldsymbol{\nu})\,\mu - \int_{\mathcal{D}} \rho \Psi(\rho^{-1}, j, \gamma)\mu.$$

• The affine Euler-Poincaré equations for $\ell$ are:

$$\begin{cases} \rho\left(\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u}\right) = \operatorname{grad}\dfrac{\partial \Psi}{\partial \rho^{-1}} - \partial_k\left(\rho\dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}_k^a}\boldsymbol{\gamma}^a\right), \\ j\dfrac{D}{Dt}\boldsymbol{\nu} - (j\boldsymbol{\nu}) \times \boldsymbol{\nu} = -\dfrac{1}{\rho}\operatorname{div}\left(\rho\dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}}\right) + \boldsymbol{\gamma}^a \times \dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}^a}, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \qquad \dfrac{D}{Dt}j + [j, \nu] = 0, \\ \dfrac{\partial}{\partial t}\gamma + \pounds_{\mathbf{u}}\gamma + \mathbf{d}^{\gamma}\nu = 0, \quad \hat{\boldsymbol{\nu}} = \nu \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3)), \end{cases}$$

where $\mathbf{d}^{\gamma}$ is the covariant $\gamma$-derivative defined by $\mathbf{d}^{\gamma}\nu(\mathbf{v}) := \mathbf{d}\nu(\mathbf{v}) + [\gamma(\mathbf{v}), \nu]$. This system recovers Eringen's equations.

The general affine Euler-Poincaré theory applied to many other complex fluids: spin chain, Yang-Mills MHD (classical and superfluid), Hall MHD, multivelocity superfluids (classical and superfluid), HBVK dynamics for superfluid $^4$He, Volovik-Dotsenko spin glasses, microfluids, Lhuillier-Rey equations (see Gay-Balmaz & Ratiu [2009]).

Kelvin-Noether circulation theorem for micropolar liquid crystals

$$\frac{d}{dt}\oint_{C_t} \mathbf{u}^\flat = \oint_{C_t} \frac{\partial \Psi}{\partial j}\mathbf{d}j + \frac{\partial \Psi}{\partial \gamma}\,\mathbf{i}\_\mathbf{d}\gamma - \frac{1}{\rho}\operatorname{div}\left(\rho \frac{\partial \Psi}{\partial \gamma}\right)\gamma .$$

The $\gamma$-circulation formulated in $\mathbb{R}^3$

$$\frac{d}{dt}\oint_{C_t} \boldsymbol{\gamma} = \oint_{C_t} \boldsymbol{\nu} \times \boldsymbol{\gamma}$$

# ERINGEN IMPLIES ERICKSEN-LESLIE

Physically, the Eringen equations should imply the Ericksen-Leslie equations. *Eringen [1993]* proposes

$$j := J(I_3 - \mathbf{n} \otimes \mathbf{n}), \qquad \gamma := \nabla \mathbf{n} \times \mathbf{n}$$

to pass from his equations to the Ericksen-Leslie equations. This is FALSE! Two arguments: brute force computation and symmetry considerations. So, one needs to do something else.

However, not all is wrong:

1. it is true that there is $\Psi(j, \gamma)$ such that

$$\Psi\left(J(I_3 - \mathbf{n} \otimes \mathbf{n}), \nabla \mathbf{n} \times \mathbf{n}\right) = F(\mathbf{n}, \nabla \mathbf{n}).$$

2. the definition $j := J(I_3 - \mathbf{n} \otimes \mathbf{n})$ is geometrically consistent.

WE SHALL USE THE TOOLS OF GEOMETRIC MECHANICS TO GIVE A DEFINITIVE ANSWER.

**Note**: For simplicity, we consider motionless nematics. The present approach easily generalizes to the flowing case.

**STEP I:** $\gamma$-formulation of Ericksen-Leslie

The material Lagrangian for nematic motionless liquid crystals $\mathcal{L} : T\mathcal{F}(\mathcal{D}, \mathrm{SO}(3)) \to \mathbb{R}$, $\mathcal{D} \subset \mathbb{R}^3$, is thus given by

$$\mathcal{L}(\chi, \dot{\chi}) = \frac{1}{2} J \int_{\mathcal{D}} \|\dot{\chi}\mathbf{n}_0\|^2 \mu - \int_{\mathcal{D}} F(\chi\mathbf{n}_0, \nabla(\chi\mathbf{n}_0))\mu,$$

where, usually $\mathbf{n}_0 = \hat{\mathbf{z}}$, $J$ is the microinertia constant, and $F$ is the Oseen-Frank free energy:

$$F(\mathbf{n}, \nabla\mathbf{n}) = K_2 \underbrace{(\mathbf{n} \cdot \operatorname{curl} \mathbf{n})}_{\textbf{chirality}} + \frac{1}{2} K_{11} \underbrace{(\operatorname{div} \mathbf{n})^2}_{\textbf{splay}} + \frac{1}{2} K_{22} \underbrace{(\mathbf{n} \cdot \operatorname{curl} \mathbf{n})^2}_{\textbf{twist}}$$
$$+ \frac{1}{2} K_{33} \underbrace{\|\mathbf{n} \times \operatorname{curl} \mathbf{n}\|^2}_{\textbf{bend}}.$$

IDEA: Apply two different EP reductions to this Lagrangian.

# FIRST EULER-POINCARÉ REDUCTION FOR NEMATICS

Write $\mathcal{L}(\chi,\dot{\chi}) = L_{\mathbf{n}_0}(\chi,\dot{\chi})$, where the Lagrangian

$$L_{\mathbf{n}_0} : T\mathcal{F}(\mathcal{D}, SO(3)) \to \mathbb{R}$$

is invariant under the right action

$$(\chi, \mathbf{n}_0) \mapsto \left(\chi\psi, \psi^{-1}\mathbf{n}_0\right)$$

of $\psi \in \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))_{\mathbf{n}_0}$ (the $G_{a_0}$ of the general theory). So get the reduced Euler-Poincaré Lagrangian

$$\ell_1(\boldsymbol{\nu}, \mathbf{n}) = \frac{1}{2}J\int_{\mathcal{D}} \|\boldsymbol{\nu} \times \mathbf{n}\|^2 \mu - \int_{\mathcal{D}} F(\mathbf{n}, \nabla\mathbf{n})\mu,$$

$\widehat{\boldsymbol{\nu}} = \dot{\chi}\chi^{-1}$, $\mathbf{n} = \chi\mathbf{n}_0$. The Euler-Poinaré equations are

$$\begin{cases} \dfrac{d}{dt}\dfrac{\delta \ell_1}{\delta \boldsymbol{\nu}} = \boldsymbol{\nu} \times \dfrac{\delta \ell_1}{\delta \boldsymbol{\nu}} + \mathbf{n} \times \dfrac{\delta \ell_1}{\delta \mathbf{n}} \\ \partial_t \mathbf{n} + \mathbf{n} \times \boldsymbol{\nu} = 0 \end{cases}$$

More explicitly, upon denoting $\mathbf{h} = -\delta\ell_1/\delta\mathbf{n}$, one has

$$\begin{cases} J\partial_t \boldsymbol{\nu} = \mathbf{h} \times \mathbf{n} \\ \partial_t \mathbf{n} + \mathbf{n} \times \boldsymbol{\nu} = 0, \end{cases}$$

which are the Ericksen-Leslie equations of nematodynamics if $\|\mathbf{n}_0\| = 1$ and $\boldsymbol{\nu}_0 \cdot \mathbf{n}_0 = 0$:

$$J\frac{d^2\mathbf{n}}{dt^2} - 2\underbrace{\left(\mathbf{n} \cdot \mathbf{h} + J\,\mathbf{n} \cdot \frac{d^2\mathbf{n}}{dt^2}\right)}_{=q}\mathbf{n} + \mathbf{h} = 0.$$

## SECOND EULER-POINCARÉ REDUCTION FOR NEMATICS

Start with the same Lagrangian. If $\mathbf{n}_0$ is constant, we can write

$$\begin{aligned}\mathcal{L}(\chi,\dot{\chi}) &= \frac{1}{2}J\int_{\mathcal{D}}\|\dot{\chi}\mathbf{n}_0\|^2\mu - \int_{\mathcal{D}} F(\chi\mathbf{n}_0, \nabla(\chi\mathbf{n}_0))\mu \\ &= \frac{1}{2}J\int_{\mathcal{D}}\|\dot{\chi}\mathbf{n}_0\|^2\mu - \int_{\mathcal{D}} F(\chi\mathbf{n}_0, (\nabla\chi)\,\chi^{-1}\cdot\chi\mathbf{n}_0))\mu,\end{aligned}$$

and we view $\mathcal{L}$ as

$$\mathcal{L}(\chi,\dot{\chi}) = L_{(\mathbf{n}_0,\gamma_0=0)}(\chi,\dot{\chi}).$$

This Lagrangian is invariant under the right action

$$(\chi,\mathbf{n}_0,\gamma_0)\mapsto\left(\chi\psi,\psi^{-1}\mathbf{n}_0,\psi^{-1}\gamma_0\psi+\psi^{-1}\nabla\psi\right)$$

of the isotropy subgroup $\mathcal{F}(\mathcal{D},SO(3))_{(\mathbf{n}_0,0)} = \mathcal{F}(\mathcal{D},S^1)\cap SO(3) = S^1$ (the $G^c_{a_0}$ of the general theory).

So we get the reduced affine Euler-Poincaré Lagrangian

$$\ell_2(\boldsymbol{\nu}, \mathbf{n}, \gamma) = \frac{1}{2} J \int_{\mathcal{D}} \|\boldsymbol{\nu} \times \mathbf{n}\|^2 \mu - \int_{\mathcal{D}} F(\mathbf{n}, -\boldsymbol{\gamma} \times \mathbf{n}) \mu.$$

$\widehat{\boldsymbol{\nu}} = \dot{\chi}\chi^{-1}$, $\mathbf{n} = \chi \mathbf{n}_0$, $\gamma = -(\nabla \chi)\, \chi^{-1} \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$.

$\rightsquigarrow$ **The correct relation bewteen $\gamma$ and $\mathbf{n}$ is $\nabla \mathbf{n} = \mathbf{n} \times \boldsymbol{\gamma}$ and not $\boldsymbol{\gamma} := \nabla \mathbf{n} \times \mathbf{n}$.**

**Notations:**

$\gamma = \widehat{\boldsymbol{\gamma}}$. For $\boldsymbol{\gamma} = \boldsymbol{\gamma}_i dx^i \in \Omega^1(\mathcal{D}; \mathbb{R}^3)$, define $\boldsymbol{\gamma} \times \mathbf{n} \in \Omega^1(\mathcal{D}, \mathbb{R}^3)$ by $\boldsymbol{\gamma} \times \mathbf{n} = \left(\boldsymbol{\gamma}_i \times \mathbf{n}\right) dx^i$, or

$$(\boldsymbol{\gamma} \times \mathbf{n})(v_x) = \boldsymbol{\gamma}(v_x) \times \mathbf{n}, \quad v_x \in T_x\mathcal{D}.$$

**Important:** $L(\chi, \dot{\chi}, \mathbf{n}_0, \gamma_0)$ may not be defined when $\gamma_0 \neq 0$. $\ell_2$ is only defined on the orbit of $\gamma_0 = 0$, i.e., if $\gamma = -(\nabla \chi)\chi^{-1}$. However, this does not affect reduction, as long as the expression $L(\chi, \dot{\chi}, \mathbf{n}_0, 0)$ is invariant under the isotropy group of $\gamma_0 = 0$. This occurs in the reduction for molecular strand dynamics with nonlocal interactions (*Ellis, Gay-Balmaz, Holm, Putkaradze, Ratiu [2010]*).

The affine Euler-Poincaré equations are

$$\begin{cases} \dfrac{d}{dt}\dfrac{\delta \ell_2}{\delta \boldsymbol{\nu}} = \boldsymbol{\nu} \times \dfrac{\delta \ell_2}{\delta \boldsymbol{\nu}} + \operatorname{div} \dfrac{\delta \ell_2}{\delta \boldsymbol{\gamma}} + \operatorname{Tr}\left( \boldsymbol{\gamma} \times \dfrac{\delta \ell_2}{\delta \boldsymbol{\gamma}} \right) + \mathbf{n} \times \dfrac{\delta \ell_2}{\delta \mathbf{n}} \\ \partial_t \mathbf{n} + \mathbf{n} \times \boldsymbol{\nu} = 0 \\ \partial_t \boldsymbol{\gamma} + \boldsymbol{\gamma} \times \boldsymbol{\nu} + \nabla \boldsymbol{\nu} = 0, \quad \boldsymbol{\gamma}_0 = 0. \end{cases}$$

If $\boldsymbol{\gamma}_0 \neq 0$, these reduced equations still make sense, and they are an extension of EL dynamics to account for disclination dynamics. Note that these equations consistently preserve the relation $\nabla \mathbf{n} = \mathbf{n} \times \boldsymbol{\gamma}$, since

$$\left( \frac{\partial}{\partial t} - \boldsymbol{\nu} \times \right) (\nabla \mathbf{n} - \mathbf{n} \times \boldsymbol{\gamma}) = 0 \,.$$

EP equations for $\ell_1$ and AEP equations $\ell_2$ are equivalent since they are induced by the SAME Euler-Lagrange equations for $\mathcal{L}(\chi, \dot{\chi})$ on $T\mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$.

Moreover, the AEP equations allow for a generalization of Ericksen-Leslie to the case with disclinations.

**STEP II:** Eringen micropolar theory contains Ericksen-Leslie director theory as a particular case

Recall:

1. Eringen's Lagrangian (motionless case, i.e., no macromotion)

$$\mathcal{L}(\chi,\dot{\chi}) = \frac{1}{2}\int_{\mathcal{D}} \operatorname{Tr}\left((i_0\chi^{-1}\dot{\chi})^T\chi^{-1}\dot{\chi}\right)\mu - \int_{\mathcal{D}} \Psi(\chi j_0\chi^{-1}, \chi\nabla\chi^{-1} + \chi\gamma_0\chi^{-1})\mu,$$

was interpreted as $\mathcal{L} = L_{(j_0,\gamma_0)}$, where $i_0 := \frac{1}{2}\operatorname{Tr}(j_0)I_3 - j_0$. This Lagrangian is invariant under the right affine action

$$(\chi, j_0, \gamma_0) \mapsto \left(\chi\psi, \psi^{-1}j_0\psi, \psi^{-1}\gamma_0\psi + \psi^{-1}\nabla\psi\right)$$

of the isotropy subgroup $\mathcal{F}(\mathcal{D}, SO(3))_{(j_0,\gamma_0)}$.

2. Reduced Lagrangian

$$\ell_2(\boldsymbol{\nu}, j, \gamma) = \frac{1}{2}\int_{\mathcal{D}} (j\boldsymbol{\nu})\cdot\boldsymbol{\nu}\mu - \int_{\mathcal{D}} \Psi(j,\boldsymbol{\gamma})\mu.$$

3. Eringen's equation are the affine Euler-Poincaré equations for:

$G = \mathcal{F}(\mathcal{D}, SO(3))$

$V^* = \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3)) \times \Omega^1(\mathcal{D}, \mathfrak{so}(3))$.

## II.1 Rod-like assumption

Take as initial condition $j_0 = J(\mathbf{I} - \mathbf{n}_0 \otimes \mathbf{n}_0)$.
This definition is $\mathcal{F}(\mathcal{D}, SO(3))$-equivariant, so that $j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$ for all time.

Consider $\mathcal{L}(\chi, \dot{\chi}) = L_{(\mathbf{n}_0, \gamma_0)}(\chi, \dot{\chi}) := L_{(j_0 = J(\mathbf{I} - \mathbf{n}_0 \otimes \mathbf{n}_0), \gamma_0)}(\chi, \dot{\chi})$. This Lagrangian is invariant under the right action

$$(\chi, \mathbf{n}_0, \gamma_0) \mapsto \left(\chi\psi, \psi^{-1}\mathbf{n}_0, \psi^{-1}\gamma_0\psi + \psi^{-1}\nabla\psi\right)$$

of the isotropy subgroup $\mathcal{F}(\mathcal{D}, SO(3))_{(\mathbf{n}_0, \gamma_0)}$.
Reduced Lagrangian

$$\begin{aligned} \ell_2'(\boldsymbol{\nu}, \mathbf{n}, \boldsymbol{\gamma}) &:= \ell_2(\boldsymbol{\nu}, J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}), \boldsymbol{\gamma}) \\ &= \frac{J}{2} \int_{\mathcal{D}} \|\boldsymbol{\nu} \times \mathbf{n}\|^2 \mu - \int_{\mathcal{D}} \Psi(J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}), \boldsymbol{\gamma})\mu, \end{aligned}$$

Affine Euler-Poincaré equations for $\ell_2'$ are equivalent to Eringen's equations in which the rod-like assumption has been assumed.

It remains to show that these equations contain, as particular case, the Ericksen-Leslie equations.

## II.2 No disclination assumption $\gamma_0 = 0$

Same step as earlier: suppose that $\mathbf{n}_0$ is constant and take $\gamma_0 = 0$.
So the evolution of $\gamma$ is given by

$$\gamma = \theta_{\chi^{-1}}(0) = -(\nabla\chi)\chi^{-1}.$$

Since $\mathbf{n} = \chi\mathbf{n}_0$, we get $\nabla\mathbf{n} = \mathbf{n} \times \boldsymbol{\gamma}$.

## II.3 Recovering the Oseen-Frank free energy

Recall that $\Psi = \Psi(j, \gamma)$, rod-like assumption $j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$, and

$$F(\mathbf{n}, \nabla\mathbf{n}) = K_2 \underbrace{(\mathbf{n} \cdot \operatorname{curl}\mathbf{n})}_{\textbf{chirality}} + \frac{1}{2}K_{11} \underbrace{(\operatorname{div}\mathbf{n})^2}_{\textbf{splay}} + \frac{1}{2}K_{22} \underbrace{(\mathbf{n} \cdot \operatorname{curl}\mathbf{n})^2}_{\textbf{twist}}$$
$$+ \frac{1}{2}K_{33} \underbrace{\|\mathbf{n} \times \operatorname{curl}\mathbf{n}\|^2}_{\textbf{bend}};$$

$K_2 \neq 0$ for cholesterics, $K_2 = 0$ for nematics.
So we need to show that there exists $\Psi = \Psi(j, \gamma)$ such that

$$\Psi(j, \boldsymbol{\gamma}) = \Psi(J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}), \gamma) = F(\mathbf{n}, \mathbf{n} \times \boldsymbol{\gamma}) = F(\mathbf{n}, \nabla\mathbf{n}).$$

**Lemma** The Oseen-Frank free energy can be expressed in terms of $\Psi = \Psi(j, \gamma)$ as

$$\Psi(j,\gamma) = \frac{K_2}{J}\operatorname{Tr}(j\boldsymbol{\gamma}) + \frac{K_{11}}{J}\left(\operatorname{Tr}\left((\boldsymbol{\gamma}^A)^2\right)(\operatorname{Tr}(j) - J) - 2\operatorname{Tr}\left(j(\boldsymbol{\gamma}^A)^2\right)\right) + \frac{1}{2}\frac{K_{22}}{J^2}\operatorname{Tr}^2(j\boldsymbol{\gamma}) - \frac{K_{33}}{J}\operatorname{Tr}\left(\left((\boldsymbol{\gamma} j)^A - J\boldsymbol{\gamma}^A\right)^2\right).$$

So we can rewrite the reduced Eringen Lagrangian in the rod-like assumption

$$\begin{aligned} \ell_2'(\boldsymbol{\nu}, \mathbf{n}, \boldsymbol{\gamma}) &= \ell_2(\boldsymbol{\nu}, J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}), \boldsymbol{\gamma}) \\ &= \frac{J}{2}\int_{\mathcal{D}} \|\boldsymbol{\nu} \times \mathbf{n}\|^2 \mu - \int_{\mathcal{D}} \Psi(J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}), \boldsymbol{\gamma})\mu \end{aligned}$$

as

$$\ell_2'(\boldsymbol{\nu}, \mathbf{n}, \boldsymbol{\gamma}) = \frac{J}{2}\int_{\mathcal{D}} \|\boldsymbol{\nu} \times \mathbf{n}\|^2 \mu - \int_{\mathcal{D}} F(\mathbf{n}, \mathbf{n} \times \boldsymbol{\gamma})\mu,$$

Same substitution in the unreduced Eringen Lagrangian in the rod-like assumption yields $\mathcal{L}(\chi, \dot{\chi}) = L_{(\mathbf{n}_0, \gamma_0 = 0)}(\chi, \dot{\chi})$.

## II.4 Recovering Ericksen-Leslie theory

1. Interpret now this $\mathcal{L}(\chi, \dot{\chi})$ as $L_{\mathbf{n}_0}(\chi, \dot{\chi})$ instead of $L_{(\mathbf{n}_0, \gamma=0)}(\chi, \dot{\chi})$.

2. Check that this Lagrangian is $\mathcal{F}(\mathcal{D}, SO(3))_{\mathbf{n}_0}$-invariant under the action $(\chi, \mathbf{n}_0) \mapsto \left(\chi\psi, \psi^{-1}\mathbf{n}_0\right)$.

3. Implement Euler-Poincaré reduction associated to the action $(\chi, \mathbf{n}_0) \mapsto \left(\chi\psi, \psi^{-1}\mathbf{n}_0\right)$ and obtain the reduced Lagrangian

$$\ell_1'(\boldsymbol{\nu}, \mathbf{n}) = \frac{J}{2}\int_{\mathcal{D}} \|\boldsymbol{\nu} \times \mathbf{n}\|^2 \mu - \int_{\mathcal{D}} F(\mathbf{n}, \nabla\mathbf{n})\mu$$

(Previously we considered affine Euler-Poincaré reduction associated to the action $(\chi, \mathbf{n}_0, \gamma_0) \mapsto \left(\chi\psi, \psi^{-1}\mathbf{n}_0, \psi^{-1}\gamma_0\psi + \psi^{-1}\nabla\psi\right)$, with reduced Lagrangian $\ell_2'$).

By general reduction theory: EP equations for $\ell_1'$ and AEP equations $\ell_2'$ are equivalent since they are induced by the SAME Euler-Lagrange equations for $\mathcal{L}(\chi, \dot{\chi})$ on $T\mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$.

It remains to show that the EP equations for $\ell'_1$ are the Ericksen-Leslie equations. True, by direct verification.

We have thus proved:

**THEOREM**: The Eringen micropolar theory of liquid crystals contains as a particular case the Ericksen-Leslie director theory. More precisely, the Ericksen-Leslie theory is recovered by assuming rod-like molecules: $j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$ and absence of disclinations $\gamma_0 = 0$.

Summary of method:

- This is shown by considering two distinct Euler-Poincaré reductions associated with distinct advected quantities.

- This allows us to replace the non-consistent definition $\gamma := \nabla\mathbf{n} \times \mathbf{n}$ by the relation $\nabla\mathbf{n} = \mathbf{n} \times \gamma$ and to solve the inconsistencies in Eringen's approach.

$$
\begin{tikzcd}
new \arrow[rr, leftrightarrow, dashed, "\nabla j = [j, \widehat{\gamma}]"] & & \textit{Eringen} \\
\textit{Ericksen-Leslie} \arrow[u, "j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})"] \arrow[d, "j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})"'] \arrow[rr, leftrightarrow, dashed, "\nabla \mathbf{n} = \mathbf{n} \times \gamma"] & & new \arrow[u, "j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})"'] \arrow[d, "j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})"] \\
\textit{Lhuillier-Rey} \arrow[rr, leftrightarrow, dashed, "\nabla \mathbf{n} = \mathbf{n} \times \gamma"] & & new
\end{tikzcd}
$$

The Lagrangians underlying the different theories. The Lagrangians on the center line identify the material descriptions of the models. The slanted arrows denote the Euler-Poincaré reduction processes while vertical arrows show how the theories are embedded in each other. By Euler-Poincaré reduction theory, all the Lagrangians related by a dashed arrow are equivalent. One of the consequences of the next is that any concrete question in a given model can be treated, equivalently, with any of the three Lagrangians in a given triangle.

$$
\begin{tikzcd}[column sep=small]
 & L(\chi, \dot{\chi}, \chi j_0 \chi^{-1}, \nabla(\chi j_0 \chi^{-1})) \arrow[dl, "j = \chi j_0 \chi^{-1}"'] \arrow[dr, "{j = \chi j_0 \chi^{-1} \atop \boldsymbol{\gamma} = -(\nabla \chi)\chi^{-1}}"] & \\
\ell_1(\boldsymbol{\nu}, j, \nabla j) \arrow[rr, dashed, leftrightarrow, "\nabla j = {[j, \widehat{\gamma}]}"] & & \ell_2(\boldsymbol{\nu}, j, \boldsymbol{\gamma}) \\
 & L(\chi, \dot{\chi}, \chi \mathbf{n}_0, \nabla(\chi \mathbf{n}_0)) \arrow[uu, "j_0 = J(\mathbf{I} - \mathbf{n}_0 \otimes \mathbf{n}_0)"'] \arrow[dl, "\mathbf{n} = \chi \mathbf{n}_0"'] \arrow[dr, "{\mathbf{n} = \chi \mathbf{n}_0 \atop \boldsymbol{\gamma} = -(\nabla \chi)\chi^{-1}}"] \arrow[dd, "j_0 = J(\mathbf{I} - \mathbf{n}_0 \otimes \mathbf{n}_0)"] & \\
\ell_1(\boldsymbol{\nu}, \mathbf{n}, \nabla \mathbf{n}) \arrow[uu, "j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})"] \arrow[dd, "j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})"'] \arrow[rr, dashed, leftrightarrow, "\nabla \mathbf{n} = \mathbf{n} \times \boldsymbol{\gamma}"] & & \ell_2(\boldsymbol{\nu}, \mathbf{n}, \boldsymbol{\gamma}) \arrow[uu, "j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})"'] \arrow[dd, "j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})"] \\
 & L(\chi, \dot{\chi}, \chi j_0 \chi^{-1}, \chi \mathbf{n}_0, \nabla(\chi \mathbf{n}_0)) \arrow[dl, "{\mathbf{n} = \chi \mathbf{n}_0 \atop j = \chi j_0 \chi^{-1}}"'] \arrow[dr, "{\mathbf{n} = \chi \mathbf{n}_0 ,\ j = \chi j_0 \chi^{-1} \atop \boldsymbol{\gamma} = -(\nabla \chi)\chi^{-1}}"] & \\
\ell_1(\boldsymbol{\nu}, j, \mathbf{n}, \nabla \mathbf{n}) \arrow[rr, dashed, leftrightarrow, "\nabla \mathbf{n} = \mathbf{n} \times \boldsymbol{\gamma}"] & & \ell_2(\boldsymbol{\nu}, j, \mathbf{n}, \boldsymbol{\gamma})
\end{tikzcd}
$$

$$\widehat{\nu} := \dot{\chi} \chi^{-1}$$

**Final remarks: 1.)** All the discussion here can be easily extended to moving liquid crystals. One applies EP, respectively affine EP, theory, as discussed earlier. Then the same considerations as above show that Eringen micropolar theory contains Ericksen-Leslie nematodynamics.

**2.)** Other inconsistencies in the micropolar description: Eringen defines a smectic liquid crystal by $\operatorname{Tr}(\gamma) = \gamma_1^1 + \gamma_2^2 + \gamma_3^3 = 0$. This is *not preserved by the evolution* $\gamma = \eta_* \left(\chi\gamma_0\chi^{-1} + \chi\nabla\chi^{-1}\right)$.

Consistent with the statement: the equation

$$\frac{\partial \gamma}{\partial t} + \pounds_{\mathbf{u}}\gamma + \mathbf{d}\nu + \gamma \times \nu = 0$$

does not imply that if $\operatorname{Tr}(\gamma_0) = 0$ then $\operatorname{Tr}(\gamma) = 0$ for all time.

Is Eringen's definition of smectic incorrect? Instead of the trace need an $\mathcal{F}(\mathcal{D}, SO(3))$-invariant function (of $\gamma$) under the action

$$\mathbf{v} \mapsto \chi^{-1}\mathbf{v} + \chi^{-1}\nabla\chi, \quad \mathbf{v} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3), \quad \chi \in \mathcal{F}(\mathcal{D}, SO(3)).$$

We do not know how to choose a physically reasonable function of this type.

**3.)** Other difficulties in liquid crystals dynamics may be solved by using the tools of geometric mechanics (disclinations, defects,...)

# ANALYTICAL RESULTS

## Esistence and uniqueness for 2D Ericksen-Leslie theory

G.A. Chechkin, T.S. Ratiu, M.R. Romanov, V.N Samokhin

Incompressible viscous 2D Ericksen-Leslie system in 3D:

$$\begin{cases} \dot{\mathbf{u}} - \mu \Delta \mathbf{u} = -\nabla p - \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial \mathbf{n}_{x_j}} \cdot \nabla \mathbf{n}\right) + \mathbf{F} + f, \quad n_{x_j} := \frac{\partial}{\partial x_j}\mathbf{n} \\ J\ddot{\mathbf{n}} - 2q\mathbf{n} + \mathbf{h} = g + \mathbf{G}, \quad \|\mathbf{n}\| = 1, \quad \operatorname{div}\mathbf{u} = 0, \quad \dot{} := \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \end{cases}$$

$\mathbf{u}$ *Eulerian* (*spatial*) *velocity*, $\mathbf{n}$ *director field*, constant $\mu > 0$ *viscosity coefficient*, constant $J > 0$ *moment of inertia of the molecule*, $\mathbf{F}(x,t)$ and $\mathbf{G}(x,t)$ given *external forces*, $f$ models dissipative part of the stress tensor, $g$ models dissipative part of intrinsic body force $f$, $g$ depend on $\mathbf{u}$, $\mathbf{n}$, and their derivatives. $F(\mathbf{n}, \nabla\mathbf{n}) =$

$$K_1 \mathbf{n} \cdot \operatorname{curl}\mathbf{n} + \frac{1}{2}\left(K_{11}(\operatorname{div}\mathbf{n})^2 + K_{22}(\mathbf{n} \cdot \operatorname{curl}\mathbf{n})^2 + K_{33}\|\mathbf{n} \times \operatorname{curl}\mathbf{n}\|^2\right).$$

*Osseen-Zöcher-Frank free energy*

Study non-dissipative regime, i.e., $g = 0$, $f = 0$. Nematic $\overset{def}{\Longrightarrow} K_1 = 0$. For simplicity, study the *one-constant approximation*, i.e.,

$$K_{11} = K_{22} = K_{33} =: K > 0.$$

With all these hypotheses, the Ericken-Leslie system becomes

$$\dot{\mathbf{u}} - \mu\Delta\mathbf{u} = -\nabla p - (K\mathbf{n}_{x_j} \cdot \nabla\mathbf{n})_{x_j} + \mathbf{F}, \quad \mathrm{div}\mathbf{u} = 0, \tag{1}$$

$$J\dot{\boldsymbol{\nu}} = -K\Delta\mathbf{n} \times \mathbf{n} + \mathbf{G}, \tag{2}$$

$$\dot{\mathbf{n}} = \boldsymbol{\nu} \times \mathbf{n}, \tag{3}$$

with unknowns $\mathbf{u}$, $\boldsymbol{\nu}$, $\mathbf{n}$. Define $\boldsymbol{\nu} = \mathbf{n} \times \dot{\mathbf{n}}$ in Ericksen-Leslie system and show directly that it implies the new first order system (1)–(3).

Conversely, if the initial conditions of the Ericksen-Leslie system satisfy the identities

$$\|\mathbf{n}(x,0)\| = 1, \quad \mathbf{n}(x,0) \perp \boldsymbol{\nu}(x,0),$$

at time $t = 0$, then for any $t > 0$ we have

$$\|\mathbf{n}\| \equiv 1, \quad \boldsymbol{\nu} = \mathbf{n} \times \dot{\mathbf{n}}, \quad 2q = \mathbf{n} \cdot \mathbf{h} - J\|\boldsymbol{\nu}\|^2,$$

and (1)–(3) turns into Ericksen-Leslie system. Thus, under these hypotheses on the initial conditions, the first order system (1), (2), (3) is equivalent to the original Ericksen-Leslie system.

Flow in $\mathbb{R}^3$ is called *two-dimensional* if all unknowns in the Ericksen-Leslie system are independent of the third coordinate $x_3$; can suppose that they are all defined on a plane $(x_1, x_2)$. Initial conditions:

$$\mathbf{u}(0,x) = \mathbf{u}_0, \quad \boldsymbol{\nu}(0,x) = \boldsymbol{\nu}_0, \quad \mathbf{n}(0,x) = \mathbf{n}_0. \tag{4}$$

Boundary conditions (if the domain is $\Omega \subset \mathbb{R}^2$):

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{n} - \boldsymbol{n}_1|_{\partial\Omega} = 0, \quad \boldsymbol{\nu}|_{\partial\Omega} = 0 \quad \text{for any } t > 0, \tag{5}$$

where $\boldsymbol{n}_1$ is a given *constant* vector field on $\Omega \times \mathbb{R}$.

Function spaces in the periodic case:

$Q_T := (0,T) \times \mathbb{T}$, $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$

$L_2(\mathbb{T}) := \left\{\mathbf{v} : \mathbb{T} \to \mathbb{R}^3 \mid \int_{\mathbb{T}} \|\mathbf{v}\|^2 d^2x < \infty\right\}$;

$W_2^m(\mathbb{T})$ is the Sobolev space of functions on $\mathbb{T}$ having $m$ distributional derivatives in $L_2(\mathbb{T})$;

$Sol(\mathbb{T}) := \{\mathbf{v} : \mathbb{T} \to \mathbb{R}^3 \mid \mathbf{v} \in C^\infty(\mathbb{T}),\ \mathrm{div}\mathbf{v} = 0\}$;

$Sol(Q_T) := \{\mathbf{v} \in C^\infty(Q_T) \mid \mathbf{v}(t,\cdot) \in Sol(\mathbb{T}),\ \forall t \in (0,T)\}$;

$Sol_2(\mathbb{T})$ is the closure of $Sol(\mathbb{T})$ in the norm $L_2(\mathbb{T})$;

$Sol_2^m(\mathbb{T})$ is the closure of $Sol(\mathbb{T})$ in the norm $W_2^m(\mathbb{T})$.

$(\mathbf{u}, \boldsymbol{\nu}, \mathbf{n}, \nabla p)$ *strong solution* of problem (1)–(4) in the domain $Q_T$ if:

**(i)** $\mathbf{u}$ time-dependent vector field in $L_2((0,T); Sol_2^2(\mathbb{T})$, $\mathbf{u}_t \in L_2(Q_T)$;

**(ii)** $\boldsymbol{\nu}$ is a vector field in $L_\infty((0,T); W_2^1(\mathbb{T}))$, $\boldsymbol{\nu}_t \in L_\infty((0,T); L_2(\mathbb{T}))$;

**(iii)** $\mathbf{n}$ is a vector field in $L_\infty((0,T); W_2^2(\mathbb{T}))$, $\mathbf{n}_t \in L_\infty((0,T); W_2^1(\mathbb{T}))$;

**(iv)** $\nabla p \in L_2(Q_T)$;

**(v)** $\mathbf{u}$, $\mathbf{n}$, $\boldsymbol{\nu}$ satisfy the initial conditions (4), i.e., $(\mathbf{u}, \mathbf{n}, \boldsymbol{\nu}) \rightharpoonup (\mathbf{u}_0, \mathbf{n}_0, \boldsymbol{\nu}_0)$ weakly in $L_2(\mathbb{T})$ as $t \to 0$;

**(vi)** equations (1)–(3) hold almost everywhere.

Function spaces in the bounded domain case:

$Q_T := (0,T) \times \Omega$, $\Omega$ Lipschitz boundary

$\overset{\circ}{Sol}\,(\Omega) := \{\mathbf{v} : \Omega \to \mathbb{R}^3$ such as $\mathbf{v} \in C_0^\infty \Omega),\ \operatorname{div} \mathbf{v} = 0\}$;

$\overset{\circ}{Sol}\,(Q_T) := \{\mathbf{v} \in C^\infty(Q_T) :\ \forall t\ \mathbf{v}(t, \cdot) \in \overset{\circ}{Sol}\,(\Omega)\}$;

$\overset{\circ}{Sol}{}_2^m\,(\Omega)$ is the closure of $\overset{\circ}{Sol}\,(\Omega)$ in the norm $W_2^m(\Omega)$

$\overset{\circ}{W}{}_2^m\,(\Omega)$ is the subspace of $W_2^m(\Omega)$ with zero trace.

$(\mathbf{u}, \boldsymbol{\nu}, \mathbf{n}, \nabla p)$ is a *strong solution* of problem (1)–(5) in $Q_T$ if:

**(i)** $\mathbf{u} \in L_2((0,T); \overset{\circ}{Sol}{}_2^1(\Omega)) \cap L_2((0,T); W_2^2(\Omega))$, $\mathbf{u}_t \in L_2(Q_T)$;

**(ii)** $\boldsymbol{\nu}$ is a vector field in $L_\infty((0,T); \overset{\circ}{W}{}_2^1(\Omega))$, $\boldsymbol{\nu}_t \in L_\infty((0,T); L_2(\Omega))$;

**(iii)** $\mathbf{n} - \boldsymbol{n}_1$ is a vector field in $L_\infty((0,T); \overset{\circ}{W}{}_2^1(\Omega)) \cap L_2((0,T); W_2^2(\Omega))$, where $\boldsymbol{n}_1$ is a given constant vector field, $\mathbf{n}_t \in L_\infty((0,T); W_2^1(\Omega))$;

**(iv)** $\nabla p \in L_2(Q_T)$;

**(v)** $\mathbf{u}$, $\mathbf{n}$, $\boldsymbol{\nu}$ satisfy initial conditions (4), i.e., $(\mathbf{u}, \mathbf{n}, \boldsymbol{\nu}) \rightharpoonup (\mathbf{u}_0, \mathbf{n}_0, \boldsymbol{\nu}_0)$ weakly in $L_2(\Omega)$;

**(vi)**equations (1)–(3) hold almost everywhere.

**Periodic case:** Let $\mathbf{F} \in L_2((0,T); W_2^1(\mathbb{T}))$, $\mathbf{G} \in L_1((0,T); W_2^2(\mathbb{T}))$; $F^3 = 0$. Suppose $\mathbf{u}_0 \in Sol_2^2(\mathbb{T})$, $\Delta \boldsymbol{\nu}_0 \in L_2(\mathbb{T})$, $\Delta \mathbf{n}_0 \in W_2^1(\mathbb{T})$. Then there is a $T > 0$ such that the solution to problem (1)–(4) exists and is unique (equality almost everywhere).

**Bounded domain case:** Suppose $\mathbf{n}_3 = 0$, i.e., $\mathbf{n} = (\cos\theta, \sin\theta, 0)$, $\boldsymbol{\nu} = (0, 0, \nu)$, $\theta$ new unknown function. Ericksen-Leslie system is:

$$\dot{\mathbf{u}} - \mu\Delta\mathbf{u} = -\nabla\left(p + \frac{K}{2}\|\nabla\theta\|^2\right) - K\Delta\theta\nabla\theta, \qquad \mathrm{div}\mathbf{u} = 0, \tag{6}$$

$$J\dot{\nu} = -K\Delta\theta, \tag{7}$$

$$\dot{\theta} = \nu \tag{8}$$

with boundary and initial conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \theta - \theta_1|_{\partial\Omega} = 0, \quad \nu|_{\partial\Omega} = 0 \quad \text{for any } t > 0, \tag{9}$$

$$\mathbf{u}(0, x) = \mathbf{u}_0(x), \quad \nu(0, x) = \nu_0(x), \quad \theta(0, x) = \theta_0(x). \tag{10}$$

$\Omega$ Lipschitz domain. For almost all $x \in \partial\Omega$, $\partial\Omega$ is graph of a $C^2$-function in some neighborhood of $x$. Let $\mathbf{F} \in L_2((0,T); W_2^1(\Omega))$, $\mathbf{G} = (0, 0, G^3) \in L_1((0,T); W_2^2(\Omega))$, $F^3 = 0$, and $\theta_0 \in W_2^3(\Omega)$, $\nu_0 \in W_2^2(\Omega)$, $\mathbf{u_0} \in \overset{\circ}{Sol}{}_2^1(\Omega) \cap W_2^2(\Omega)$. Let $\Delta\mathbf{u}_0|_{\partial\Omega} = 0$ and assume that for some $d > 0$ we have

$$\theta_0(x) = \theta_1 \equiv const, \quad \nu_0(x) = 0 \quad \text{if } dist(x, \partial\Omega) < d.$$

Then the solution to (6)–(10) exists for some $T > 0$ and is unique.

## Finite propagation speed

The following theorem is true in both the periodic and the bounded domain case.

Consider the equations (2)–(4) and suppose that $w^{ij} := u^j_{x_i} + u^i_{x_j}$ for some $1 < \alpha \leq \infty$ satisfy

$$\|\text{esssup}_x |w^{ij}(x,t)|\, \|_{L_\alpha(0,T)} \leq M/d$$

and $\|\mathbf{u}\|$ is bounded by a constant $m$. Assume also that $\nabla \mathbf{n}_0$ and $\boldsymbol{\nu}_0$ vanish for $\|x - x_0\| < r$. Then $\nabla \mathbf{n}$ and $\boldsymbol{\nu}$ are equal to zero for

$$\|x - x_0\| < r - (m + \max\{1, K/J\})t, \quad Mt^{\frac{\alpha}{\alpha-1}} \leq \frac{1}{2}.$$

## Blow up of smooth solutions for isentropic viscous liquid crystals

T.S. Ratiu and Olga Rozanova

The full 3D equations of isentropic viscous liquid crystals are:

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \tag{11}$$

$$\partial_t(\rho \mathbf{u}) + \mathrm{Div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathrm{Div}(\sigma^R) + \mathrm{Div}(\sigma^D), \tag{12}$$

$$\frac{D}{Dt} n_i = \Omega_{ki} n_k + \lambda(\delta_{il} - n_i n_l) d_k A_{kl} + \frac{h_i}{\beta}, \tag{13}$$

$$p = p(\rho), \tag{14}$$

$$(\rho, \mathbf{u}, \mathbf{n})\Big|_{t=0} = (\rho_0, \mathbf{u}_0, \mathbf{n}_0), \quad \mathbf{n}_0 \in S^2 \tag{15}$$

Equations are in $\mathbb{R} \times \mathbb{R}^3$ (Landau-Lifshitz, Vol 7, 3rd ed.); conservation of mass, linear momentum, and $\mathbf{n} \in S^2$. $\rho$ is the fluid density, $p$ is a pressure, $\mathbf{u} = (u_1, u_2, u_3)^T$ is the spatial (Eulerian) velocity, $\mathbf{n} = (n_1, n_2, n_3)^T$ is the orientational order parameter representing the macroscopic average of the molecular directors.

$\frac{D}{Dt} := \frac{\partial}{\partial t} + (u, \nabla)$ is the material derivative
$\mathrm{div(a)} := \frac{\partial \mathrm{a_i}}{\partial \mathrm{x_i}}$ is the usual divergence of a vector field $(a_1, a_2, a_3)^T$
$\mathrm{Div}(\tau)_{\mathrm{i}} := \frac{\partial \tau_{\mathrm{ij}}}{\partial \mathrm{x_j}}$ is the divergence of a 2-tensor $(\tau_{ij})$
$A := \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the symmetric part of strain rate
$\Omega := \frac{1}{2}(\nabla u - (\nabla u)^T) = \mathrm{curl}\, u$ is the skew-symmetric part of the strain rate; vorticity.
Oseen-Zöcher-Frank free energy the sum of the splay, the bend, and the twist, i.e,

$$W(\rho, d) = K_1 \frac{1}{2}(\mathrm{div}\, \mathbf{n})^2 + K_2 \frac{1}{2} \|\mathbf{n} \times (\mathrm{curl}\, \mathbf{n})\|^2 + K_3 \frac{1}{2}(\mathbf{n} \cdot \mathrm{curl}\, \mathbf{n})^2 \geq 0,$$

$\lambda, \beta \in \mathbb{R}$ are constants, $K_i \geq 0,\ i = 1, 2, 3$, are functions of $\rho$. The vector field $h$ in (13) is defined by

$$h := H - (\mathbf{n}, H)\mathbf{n}, \quad H_i := \partial_k \pi_{ki} - \frac{\partial W}{\partial n_i}, \quad \pi_{ki} := \frac{\partial W}{\partial(\partial_k n_i)}.$$

Let us note that (13) implies $\|\mathbf{n}\| = 1$ for $t > 0$ if $\|\mathbf{n}_0\| = 1$.

Reactive (non-dissipative) symmetrized part of the stress tensor:

$$\sigma_{ik}^{R} = -\frac{1}{2}(n_i h_k + n_k h_i) - \frac{1}{2}(\pi_{kl}\partial_i n_l + \pi_{il}\partial_k n_l) - \frac{1}{2}\partial_l[(\pi_{ik} - \pi_{ki})n_l - \pi_{kl}n_i - \pi_{il}n_k].$$

Dissipative symmetrized part of the stress tensor:

$$\sigma_{ij}^{D} = \mu_1 A_{ij} + \mu_2(A_{ik}d_k d_j + A_{jk}d_i d_k) + \mu_3\delta_{ij}A_{kk}$$
$$+\mu_4 d_i d_j d_k d_l A_{kl} + \mu_5[\delta_{ij}d_k d_l A_{kl} + d_i d_j A_{kk}].$$

Coefficients $\mu_1, ..., \mu_5$ may depend on $\rho$.

Total mass: $m(t) = \int\limits_{\mathbb{R}^3} \rho \, dx,$

Total linear momentum: $P(t) = \int\limits_{\mathbb{R}^3} \rho u \, dx,$

Total energy: $E(t) = \frac{1}{2}\int\limits_{\mathbb{R}^3} \rho\|\mathbf{u}\|^2 \, dx + \int\limits_{\mathbb{R}^3} \Psi(\rho) \, dx + \int\limits_{\mathbb{R}^3} W(\rho, \mathbf{n}) \, dx = E_k(t) + E_i(t) +$

$E_d(t) \geq 0$, kinetic, internal, deformational energy. $\Psi(\rho) = \int\limits_0^{\rho} \frac{p'(\xi)}{\xi} \, d\xi \geq 0.$

Assume that there exists $\gamma > 1$ such that

$$\Psi(\rho) \geq A\rho^{\gamma}, \quad A = \text{const} > 0. \qquad (16)$$

A solution $(\rho, \mathbf{u}, \mathbf{n})$ to the Cauchy problem (11)–(15) belongs to the class $\mathfrak{K}$ if the solution is classical, $\rho \geq 0$, $\mathbf{u} \in H^1(\mathbb{R}^3)$, the mass $m(t)$, linear momentum $P(t)$, and total energy are finite for all $t \geq 0$ for which the solution exists, and, $m(t) = m = \mathrm{const}$, $P(t) = P = \mathrm{const}$.

Thus, if the solution belongs to the class $\mathfrak{K}$, then

$$\rho \in L_1(\mathbb{R}^3), \quad \Psi(\rho) \in L_1(\mathbb{R}^3), \quad \sqrt{\rho}\mathbf{u} \in L_2(\mathbb{R}^3), \quad W \in L_1(\mathbb{R}^3).$$

Energy decay along smooth solutions

$$\frac{d}{dt}E(t) = -\int_{\mathbb{R}^3} \left( \sigma_{ik}^D A_{ik} + \frac{\|h\|^2}{\beta} \right) dx.$$

Assume the following inequalities

$$\beta > 0, \quad \mu_1 > 0, \quad \mu_1 + \mu_3 \geq 0, \quad \mu_4 \geq 0, \quad \theta := \frac{\mu_1}{2} - 2|\mu_2| - 4|\mu_5| > 0. \tag{17}$$

Then there exists a constant $\theta > 0$ such that

$$\frac{d}{dt}E(t) \leq -\theta \int_{\mathbb{R}^3} |Du|^2 \, dx.$$

Assume that for $\gamma \geq \frac{6}{5}$ we have $\Psi(\rho) \geq A\rho^{\gamma}$ for some constant $A > 0$. If $\|P\| \neq 0$, then there exists $C > 0$ such that for the solutions from the class $\mathfrak{K}$ the following inequality holds:

$$\int_{\mathbb{R}^3} |Du|^2\, dx \geq C\, E_i(t)^{-\frac{1}{3(\gamma-1)}}.$$

This leads to the following theorem:

Suppose $\gamma \geq \frac{6}{5}$, $\|P\| \neq 0$, and inequalities (16), (17). Then there is no global in time solution to the Cauchy problem (11)–(15) in the class $\mathfrak{K}$.

The proof shows that the blow-up is due to the presence of the viscosity term. The nature of loss of smoothness is similar to the case of the compressible Navier-Stokes equation and the anisotropic features do not influence this phenomenon. Thus, the addition of liquid crystal degrees of freedom does not regularize the Navier-Stokes equations. From experimental data it seems that blow-up is related to high temperature.